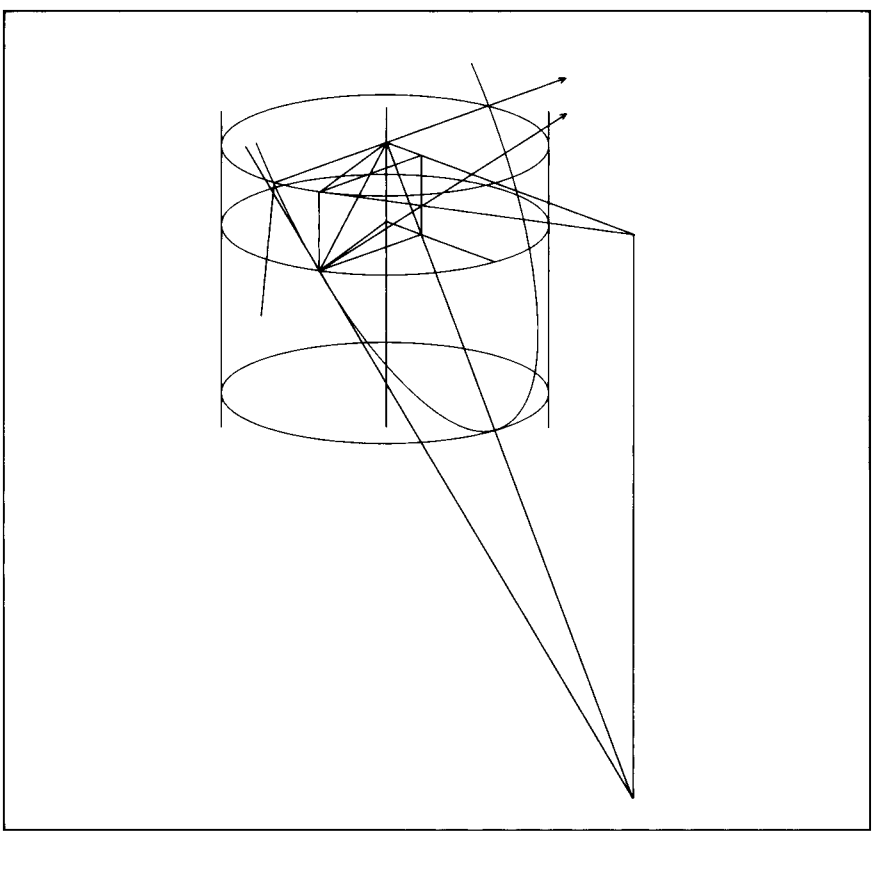
ALHACEN ON IMAGE-FORMATION AND DISTORTION IN MIRRORS

A Critical Edition, with English Translation and Commentary, of Book 6 of Alhacen's *De Aspectibus*, the Medieval Latin Version of Ibn al-Haytham's *Kitāb al-Manāẓir*

Volume Two

English Translation



A. Mark Smith

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VOLUME ONE
Introduction and Latin Text

VOLUME TWO
English Translation

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CONTENTS

VOLUME II

BOOK SIX OF ALHACEN'S *DE ASPECTIBUS*

Topical Synopsis

[1.1] The agenda for book 6: explaining image-distortion in the seven baisc types of mirrors.

[2.1-2] Just as in direct vision, so in reflected vision, misperception can arise from an excess or defect in the threshold conditions governing veridical vision. Misperceptions in reflected vision are magnified by the weakening of light caused by reflection itself. [2.3] In reflected vision misperception is exacerbated by three factors: first, the object appears to lie where it actually does not; second, the object's color mingles with that of mirror; third, the object's light and color appear more dimly because of the weakening caused by reflection. [2.4] Because of this weakening, the limits for the threshold of veridical vision are truncated in reflected vision.

[3.1] Introductory paragraph. [3.2-4] PROPOSITION 1: The image in a plane mirror lies the same distance below the reflecting surface as its object, and it is the same size and shape as its object. [3.5-6] Because of the weakening of light and color due to reflection, a viewer can misperceive light, color, number, and distance in ways that would not occur were the object viewed directly. [3.7] Image-reversal, which involves spatial disposition, is a misperception unique to reflection. [3.8] Reflection invariably causes misperception of light, color, and spatial disposition; misperceptions stemming from these misperceptions are like

those in direct vision, only they occur more easily. **[3.9]** Reflected vision is subject to diplopia in the same way as direct vision. **[3.10]** Images seen in plane mirrors appear to lie even farther away than they would if they were objects lying at the same distance but seen directly. **[3.11]** Misperception of shape can occur in reflected vision for the same reason as in direct vision, but that misperception is more frequent and more exaggerated in reflected vision. **[3.12]** Misperception of disjunction or separation can occur in reflected vision as it can in direct vision.

[4.1] Introductory statement: Aside from the misperceptions these mirrors have in common with plane mirrors, they also diminish the size of objects appearing in them; the only thing about the object that appears unmodified in these mirrors is the arrangement of its parts. **[4.2-10]** PROPOSITION 2: An object should always appear smaller than it is in a convex spherical mirror. **[4.11-71]** PROPOSITION 3: It is possible for the image to appear the same size as, or larger than, its object in such a mirror. **[4.72]** Transitional paragraph. **[4.73-79]** PROPOSITION 4, LEMMA 1: If two points that are equidistant from the mirror's center lie different distances from the center of sight, then the image of the point lying farther from the center of sight will lie farther from the mirror's center than the image of the point lying nearer the center of sight, and the endpoint of tangency for the farther point will lie farther from the mirror's center than the endpoint of tangency for the nearer point, no matter whether these points lie in the same plane as the center of sight or not. **[4.80-85]** PROPOSITION 5, LEMMA 2: Given a line AB cut at points G and D so that AB:BD = AG:GD, if three lines are erected to points B, G, and D so as to intersect above line AB, and if a line is drawn through them from point A, that line will be cut in the same ratio as line AB. **[4.86-88]** PROPOSITION 6, LEMMA 3: Given line AB cut at points G and D so that AB:BD = AG:GD, if a line is drawn from A and cut according to the same ratio, and if lines are drawn through the respective points of division on both lines, those lines will intersect at a point. **[4.89-90]** PROPOSITION 7, LEMMA 4: Given line AB cut at points G and D so that AB:BD = AG:GD, if parallel lines are erected to that line from points B, G, and D, and if a line is passed through them from A, that line will be cut in the same ratio as AB. **[4.91-101]** PROPOSITION 8: In convex spherical mirrors the curvature of the image of an arc accords with

the mirror's center, not with its surface. **[4.101-104]** PROPOSITION 9: The more sharply curved a line is, the less sharply curved it may appear in such a mirror. **[4.105-107]** PROPOSITION 10: The images of straight lines whose endpoints are equidistant from the center of curvature appear curved in such a mirror. **[4.108-110]** PROPOSITION 11: The images of straight lines whose endpoints are not equidistant from the center of curvature are curved when those lines or their extensions do not touch the mirror's surface. **[4.111-120]** PROPOSITION 12: If a straight line or its extension is tangent to the mirror, its image will be curved. **[4.121-125]** PROPOSITION 13: The image of a straight line that intersects the mirror's surface will be curved. **[4.126-140]** PROPOSITION 14: If a straight line lies in the same plane as the center of sight, then, depending on how it is poised with respect to the mirror's surface and the center of sight, all, some, or none of that line may be seen in the mirror. **[4.141-152]** PROPOSITION 15: When the visible line, the center of sight, and the center of curvature all lie in the same plane, the image of that line will appear curved. **[4.153-154]** Summary of conclusions for chapter 4.

[5.1] Introductory statement: in convex cylindrical mirrors the same misperceptions occur as occur in convex spherical mirrors, except they are more pronounced in convex cylindrical mirrors. **[5.2]** The agenda for this chapter according to whether the plane of reflection forms a line of longitude, a circle, or an ellipse on the cylinder's surface. **[5.3-11]** PROPOSITION 16, LEMMA 5: In an elliptical section formed on a convex cylindrical mirror, if a point is chosen on it that is not a point of reflection and if a line is drawn from that point to intersect the normal dropped from the point of reflection at the cylinder's axis, then when a another line is dropped through that point perpendicular to the tangent at that point, this perpendicular will intersect the normal dropped from the point of reflection beyond the cylinder's axis and beyond the intersection of the first line with the normal dropped from the point of reflection. **[5.12-20]** PROPOSITION 17: If a straight line lies outside a convex cylindrical mirror within a plane passing through a line of longitude on the cylinder and its axis, its form will be reflected from that line of longitude, and if the center of sight lies within that same plane, the image will be a straight line. **[5.21-27]** PROPOSITION 18: Given the previous situation, if the center of sight does not lie in the same

BOOK SIX

This book is divided into nine chapters. The first chapter [describes] the basic purport of the book; the second [explains] that error occurs in sight because of reflection; the third [focuses] on error that arises in plane mirrors; the fourth [focuses] on error that originates in convex spherical mirrors; the fifth [focuses] on convex cylindrical mirrors; the sixth [focuses] on convex conical [mirrors]; the seventh [focuses] on concave spherical [mirrors]; the eighth [focuses] on concave cylindrical [mirrors]; the ninth [focuses] on concave conical [mirrors].

CHAPTER 1

[1.1] It was shown in the preceding books how forms are apprehended in mirrors by the visual faculty, how the lines of reflection or incidence are disposed, [and] how images are disposed and where they are located. However, the form is not always perceived as it actually exists by means of reflection. For in concave mirrors the image of [one's] face appears distorted, and its proper disposition is obscured from sight, so it is obvious that error occurs in the perception of forms through reflection. In the present book it is [our] purpose to explain how this error occurs and the reason for it, as well as to discuss the different types of errors due to the different types of mirrors.

CHAPTER 2

[2.1] The second book showed how forms are perceived in direct vision, and the third book carefully analyzed the particular factors that lead to error in that [kind of] vision when the [conditions for proper vision] exceed or fall short of the [appropriate] threshold. The perception of forms by means of a reflection [of rays] occurs in the same way [as it does] in direct vision, and [so] the things that are apprehended in direct vision are also apprehended in reflected vision—such things as light, color, shape, size, distance, and the like [i.e., the full range of visible intentions].

[2.2] Moreover, just as happens in the direct visual apprehension of things [whose forms] are already ensconced [in the soul] and known, so in reflected vision there is a correlation [of the form] to something else [like it] so that a conclusion is drawn and a judgment is made in the soul. Hence, any excesses or defects in the threshold conditions [for proper sight that] cause an error in direct vision likewise cause [an error] in reflected vision. And according to each case [of excess or defect in the threshold condition], the error is magnified in reflected vision because of the diminished light that results from the weakening caused by the actual reflection.

[2.3] Furthermore, to generalize, we should say that the proper disposition of the form cannot be perceived in reflected vision as it can be in direct vision because of a threefold constraint specific to reflection. The first is that in reflection the form of the object appears to the viewer to lie directly in front of the eyes when this is not actually the case. The second [is] that the light and color in the visible object are mingled with the color of the mirror, and the visual faculty perceives that mingled [color] rather than the actual color or light belonging to the visible object. The third is that, as has been pointed out earlier [in book 4], reflection itself weakens light and color, so the actual light and color will be less clearly seen in reflected vision than in direct vision.

[2.4] In addition, earlier discussions showed that the range of the [limits of the] threshold conditions [whose excess or defect] leads to error depends on the intensity of the light and color, for that range will be greater in stronger light or color [and] less in weaker [light or color]. And, since light and color will be weakened by reflection, the range of the [limits of the] threshold conditions [whose excess or defect] leads to particular kinds of error will be less in reflected vision than in direct vision, and the shortening of that range leads to an increase in the number of errors. Besides, certain tiny features of objects can be perceived through direct vision that are in no way perceptible through reflected vision. It is therefore evident that reflected vision exceeds direct vision in the degree and number of errors.

CHAPTER 3
[On Plane Mirrors]

[3.1] In each kind of mirror a misperception of forms occurs, but the variety of errors [that occur] depends on the variety of mirrors [in which the forms are perceived]. In plane mirrors less error occurs than in the others. For in these [kinds of mirrors] the proper shape, spatial disposition and size [of the object] are perceived, just as [they are] in direct vision, which will be shown by [the following] demonstration.

[3.2] **[PROPOSITION 1]** Imagine a plane mirror, and let line AB [in figure 6.3.1, p. 97] on that mirror's surface be the common section of the mirror's surface and a plane perpendicular to the mirror's surface. Let H and Z be two points in that perpendicular plane, [let] E [be] the center of sight, and draw perpendicular HL from point H to the mirror's surface. Extend it so that LG = LH. Likewise, extend perpendicular ZF so that DF = FZ.

[3.3] It is clear from earlier discussions [i.e., book 5, proposition 1, in Smith, *Alhacen on the Principles*, 399] that [the form of point] H is reflected to [point] E from a point on the mirror, and its image-location G lies as far from the mirror's surface [below it] as H [lies above it]. By the same token, [the form of point] Z is reflected to [point] E, and its image-location is D.

[3.4] Now when line ZH is drawn, and likewise line GD, [the form of] any point on line ZH is reflected to [point] E. Its image-location lies the same distance from the mirror's surface as the point itself, and so any point on line ZH appears to lie the same distance [from the mirror's surface] as it will [actually] lie [from that surface]. Hence, if line ZH is straight, line DG will be straight. If it is curved, DG will be an arc of the same curvature, so line ZH will appear the same size and shape as it is, which is what was set out [to be proven].

[3.5] However, if there are various colors that are only slightly different from one another at points along line ZH, the variation [among them] may not be perceived; instead, a single blend of color will be presented to sight. Hence, because of reflection there will be an error involving light and color, and in addition [an error] concerning number. For that difference among the colors and lights might be perceptible in direct vision, but the [perceptibility of the] color has exceeded the threshold with respect to reflected vision, although not with respect to direct vision. Likewise, tiny features that could be discerned in direct vision are either hidden or confused in reflected vision.

[3.6] Moreover, because of the weakening of light or color by reflection, an error arises in [perception] of distance that would not arise in direct vision.[1]

[3.7] In the case of spatial disposition an error clearly arises from reflection alone, for in the image we perceive things on the left-hand side of the visible object that we would see on the right-hand side if the object were [actually placed in front of us] at the image-location. For, when something faces something else, its corresponding spatial disposition is opposite because what is the right-hand side of the one will be the left-hand side of the other. Accordingly, the right-hand side of the visible object is the left-hand side of the image, whereas the left-hand side of the image will be its

right-hand side to the viewer, but it is perceived on the left-hand side of the image.[2]

[3.8] Overall, in the case of light, color, or spatial disposition, error invariably arises from the very reflection itself. In these cases, as well as in others, the things that lead to error in direct vision likewise lead to error in reflected vision, and more easily because the [range of] threshold conditions for each is smaller in reflected vision than in direct vision. One example for all of these [cases] may be applied, and the same should be understood [to apply] to the rest.

[3.9] In direct vision, when the visible object lies far outside the visual axes, it may appear double; the same thing happens in mirrors when the visible object lies far outside the visual axes.

[3.10] In mirrors, the object will appear smaller than it should at a given distance, whereas at such a distance it may look smaller than it should in direct vision, but not to such a great extent.[3] And this increased diminution [which happens] in mirrors is due to the decrease in the [range of] threshold conditions [for the perception] of distance.

[3.11] In [the perception of] shape error sometimes arises in mirrors for the same reasons it does in direct vision, but [it does so] more significantly and more frequently according to spatial disposition.

[3.12] If a rope or something like it faces a mirror at a given distance, and if its ends cannot be perceived by the visual faculty, it may appear to lie on the very surface of the mirror. The same thing happens in direct vision. If some rope is placed facing a window and the ends of the rope cannot be seen, the separation between rope and window will not be apparent, even if it is significant, and [this] is due to spatial disposition.[4] Moreover, if one of the ends is visible but the other is not, that end may appear to lie in the plane [of the window]. In each case, where [error] occurs in direct vision, it occurs likewise in reflected vision.

CHAPTER 4
On [Convex] Spherical Mirrors

[4.1] The entire range of errors that occur in plane mirrors also occurs in convex spherical [mirrors], and besides this, a visible object looks smaller than it should in [convex] spherical mirrors. Overall, in these [kinds of] mirrors nothing about the visible object is perceived as it actually is except the arrangement of its parts, which appears in the mirror as it actually exists in the visible object.

[4.2] **[PROPOSITION 2]** That an object should always appear smaller than it is in this [sort of] mirror is demonstrated [as follows].

[4.3] Let AB [in figure 6.4.2, p. 97] represent a visible line [on some object, let] ZP be the mirror, D the center of the [great] circle [produced by the plane of reflection on the mirror], and E the center of sight. Let [the form of point] A be reflected to [point] E from point H, and [let the form of point] B [be reflected to E] from point N. When it is extended, line AB will pass through the center of the mirror, or [it will] not.

[CASE 1]

[4.4] Let it pass through. From point N draw line NL tangent to the circle, and from point H [draw] tangent HM. Draw the line[-couple]s of reflection BN and EN, and AH and EH, extend lines EH and EN until they fall on normal AD, and let T and Q be the points where they fall. It is evident that T is the image-location for A, and Q is the image-location for B. I say that AB > QT.

[4.5] It is evident from previous discussions [in book 5, prop. 7] that AD:DT = AM:MT. Likewise, BD:DQ = BL:LQ.[5] But AD > BD, and DT < DQ [so AD:DT > BD:DQ and thus > BL:LQ]. Hence, AM:MT [which = AD:DT] > BL:LQ.

[4.6] Cut AM at point F so that FM:MT = BL:LQ. Therefore, BM:MT < BL:LQ. Cut MT at point K so that BM:MK = BL:LQ. K will necessarily fall between M and Q because LQ < MQ, and BL > BM. Accordingly, since FM: MT = BL:LQ, as well as BM:MK, FB:KT = BL:LQ.[6] But BL > LQ [because we know by previous conclusions that BD:DQ = BL:LQ, and BD > DQ, so BL > LQ]. Hence, FB > KT, so [visible line] AB > [its image] QT [since AB > FB, and QT < KT], which is what was set out [to be proven].

[CASE 2]

[4.7] But if line AB, when it is extended, does not reach the center [of the circle], then from point A [in figures 6.4.2a and 6.4.2b, p. 98] draw line AG to the center, let G be the center, and from point B draw line BG. Let point D be the image-location for [point] A, let [point] E be the image-location for [point] B, and draw line ED, which is the image of line AB. I say that [object] AB > [image] ED because ED is either parallel to AB or not.

[4.8] If it is parallel [as in figure 6.4.2a], it is clear that it is smaller [i.e., ED < AB because triangles EDG and BAG are similar, so BA:ED = BG:EG, and BG > EG]. If it is not parallel [as in figure 6.4.2b], extend [ED] until it meets AB. Let Z be the [point of] intersection, and from point E draw EH parallel to AB. Angle EDH is acute, right, or greater [than a right angle].

[4.9] If it is right or greater [than a right angle], side EH > [side] ED. But [by previous conclusions] EH < AB, and so [we have demonstrated] what was set out [to be proven].[7]

[4.10] If it is acute, it could happen that the form [i.e., ED] is larger than the object [AB] whose form it is, which, although it may be larger, will happen rarely. And when it does happen, the form may be perceived from such a distance that it will appear smaller than it should because the object itself may appear smaller [than it should] at that distance [in direct vision].[8]

[4.11] **[PROPOSITION 3]** It will now be demonstrated that in these [kinds of] mirrors a form may sometimes appear larger than the visible object, i.e., when it [actually] is larger, and that it may be perceived [as larger] from such a distance that its size can be discerned with proper certitude.

[4.12] Let A [in figure 6.4.3, p. 99] be the center of the mirror, and take a plane of reflection that will cut the mirror along a [great] circle. Let that circle be EDB, let ED be the diameter of that circle, and extend diameter ED to Z so that rectangle EZ,ZD is not greater than AD^2, which is clear[ly possible], since it is possible for a line to be added to diameter ED such that the rectangle formed by the whole and the added part equals AD^2 [by Euclid, III.36].[9] Bisect line ZD at point H. Hence, AH will be half of EZ. Accordingly, [since AD < AH, which = half EZ, while DH = half DZ] rectangle AD,HD will not be greater than one-fourth AD^2 [because it is no greater than one-fourth rectangle EZ,ZD, which is no greater than AD^2], and since AH,HD > HD^2, let AH,HT = HD^2.[10]

[4.13] Produce a circle according to length AH [as radius], and from point H draw chord HQ equal to one-half line HD. Draw lines QA and QT, and at point Q form angle HQN equal to angle QAH. Accordingly, since these two angles in these two triangles [HQN and QAH] are equal, and since one [angle], i.e., QHA, is common, the third [angle] = the third [angle], i.e., [angle] AQH = angle HNQ. And [so] the triangles will be similar [by Euclid, VI.4], and [according to proportional sides] AH:HQ = HQ:HN. Therefore, AH,HN = HQ^2 [by Euclid, VI.17].

[4.14] But HQ^2 is one-fourth HD^2, since HQ is one-half HD [by construction]. Therefore, AH,HN = one-fourth AH,HT [which = HD^2, by construction], so HN is one-fourth HT. Accordingly, N lies between H and T. It follows that HT,TN is three-fourths HT^2.

[4.15] Angle QHD is acute, however, and [it is] equal to angle HQA, since they are subtended by equal sides in the larger triangle [i.e., QA and HA, which are radii of the larger circle]. Therefore, angle QHN [in triangle AQH] = angle HNQ [in triangle HQN, which is similar to triangle AQH, by previous conclusions], and so HQ = QN.

[4.16] Also, angle HNQ is acute, so [adjacent] angle QNT is obtuse. Hence, TQ^2 exceeds QN^2 + TN^2 by TN,NH because, as Euclid claims [in

II.12], the square on the opposite side of an obtuse [angle] exceeds the squares on the [other] two sides by twice the rectangle formed by one of the sides and the adjoining segment that extends to where the perpendicular is dropped [to it] from the endpoint of the other side. And if a perpendicular is dropped from point Q to line HT, it will fall at the midpoint of line HN [because triangle QNT is isosceles], and twice the rectangle formed by TN and one-half HN equals TN,HN.

[4.17] Therefore, TQ^2 exceeds $QN^2 + TN^2$ by TN,NH [i.e., $TQ^2 - QN^2 - TN^2$ = NH,NT, so NH,NT + $TN^2 = TQ^2 - QN^2$]. But HN,NT + NT^2 = HT,TN [by Euclid, II.3]. Therefore HT,TN = $TQ^2 - HQ^2$ [because $HQ^2 = QN^2$, since HQ = QN in isosceles triangle QHN].

[4.18] Now let AI:AH = QT:QH [by Euclid VI.12]. The square [of AI] to the square [of AH] will be as the square [of QT] to the square [of QH—i.e., $AI^2:AH^2 = QT^2:QH^2$], and $(AI^2 - AH^2):AH^2$ = HT,TN [which = $QT^2 - QH^2$]: QH^2 [by Euclid, V.17]. And since $4QH^2 = HD^2$ [by previous conclusions], while 4HT,TN = $3HT^2$ [by previous conclusions], HT,TN:QH^2 = $3HT^2$ [which = 4HT,TN]:HD^2 [which = $4QH^2$].

[4.19] Moreover, let HC = 3HT. [Therefore,] CH,HA = $3HD^2$ [since HT,HA = HD^2, by construction], but since AH:HD = HD:HT [because AH,HT = HD^2, by construction, so HD is the mean proportional between AH and HT], HT:HA = $HT^2:HD^2$.[11] But CH:HA = CH,HT:HA,HT, and so CH:HA = $3HT^2:HD^2$ [because CH:HA = 3HT:HA]. But this [i.e., $3HT^2:HD^2$] was as $(AI^2 - AH^2):AH^2$.[12] Therefore, CH:HA = $(AI^2 - AH^2):AH^2$. Therefore, CA [which = CH + HA]:AH = $AI^2:HA^2$, for $(AI^2 - HA^2) + HA^2 = AI^2$.

[4.20] Hence, IA will be the mean proportional between CA and HA, whose converse we touched upon a bit earlier.[13] Accordingly, CA:IA = IA: HA, and the remainder will be in the same proportion to the remainder, i.e., CI:IH [= CA:IA = IA:HA], and since IA > HA, CI > IH.

[4.21] Furthermore, AH,HD < one-fourth AD^2 [by previous conclusions]. Therefore, [line] HD < one-fourth line AD [because AH > AD]. So it is less than one-fifth AH [because AH = AD + HD]. Therefore, since AH > 5HD, and since AH,HT = HD^2 [by construction], HT < one-fifth HD, and so HT < one twenty-fifth HA. But, as was [just] claimed, CI:IH = IA:HA. Thus, CH [which = CI + IH]:IH = (IA + AH):AH. Hence, one-third the first [term is] to the second as one-third the third [term is] to the fourth [i.e. one-third CH:IH = one-third (IA + AH):AH].

[4.22] But [line] HT is one-third line CH [by construction]. Therefore, TH:IH = one-third (line IA + AH):line AH. Accordingly, TH:IH = (two-thirds line AH + one-third line IH):line AH.[14] However, since CI > IH [from previous conclusions], IH < one-half CH, and one-third IH < one-sixth CH, and so one-third IH < one-half TH [which = one-third CH]. Therefore,

(two-thirds AH + less than one-half HT):AH = TH:IH. So, conversely, IH:HT = AH:(two-thirds AH + less than half HT).

[4.23] But HT < one twenty-fifth AH [by previous conclusions], and its half < half of one twenty-fifth. But line AH is divided into twenty-five parts, [so] two-thirds [of those 25 parts, i.e., < 17] + half of one twenty-fifth [part] does not add up to 18 parts. Therefore, IH:HT > 25:18.

[4.24] Furthermore, since HT < one twenty-fifth AH, AT > twenty-four twenty-fifths AH. But line IH < one-half CH, so [IH] < (HT + one-half HT), so [IH] < one and one-half of [one of] the twenty-five parts comprising AH, and so IA < twenty-six-and-one-half of the given 25 parts into which HA is divided. Therefore, IA:AT = (less than twenty-six-and-a-half):more than 24. Thus, IA:AT < twenty-six-and-a-half:24. But IH:HT > 25:18. Therefore, IH:HT > IA:AT.

[4.25] [By Euclid VI.12] let IM:MT = IA:AT [in figure 6.4.3a, p. 100]. M will therefore fall between I and H. Moreover, IM:MH > IA:AT, and so it is greater than IA:AH. So let IL:LH = IA:AH [by Euclid VI.12]. L will of course fall between M and I.

[4.26] Now from points L and M draw tangents LB and MG, and draw lines IB, HB, IG, TG, AB, and AG, and extend the last [two] to the outer circle [to intersect it at points Z_2 and Z_1, respectively].[15]

[4.27] From the fifth [proposition] of the fifth book you will [then] have that angle IBZ_2 = angle HBA, for, since IL:HL = IA:AH [by construction], H will be the image-location [of object-point I] in the case of reflection from point B [when HB, extended beyond B to H′, forms the line of reflection].[16] And if the contrary is claimed, so that some other image-location is chosen, you will disprove it by a *reductio ad absurdum,* given that it is impossible for the ratio of IA to the line from point A to the image-location not to be as [the ratio of] IL to the line from point L to the image-location.

[4.28] Therefore, since H is the image-location, and since LB is tangent [to the mirror] on AB, then when it is extended [to H′], HB will form an angle of reflection [H′BZ_2] equal to its vertical [angle HBA], and because LB is perpendicular to ABZ_2, it will follow that angle IBL = angle LBH. By the same token, angle IGZ_1 = angle TGA, and since MG is perpendicular [to AG], angle IGM = angle MGT.

[4.29] Now draw line HP from point H to line AB parallel to [line] IB, and from point T [draw] TR parallel to IG. Angle IBZ_2 = angle HPB. But, as was claimed [earlier], angle IBZ_2 = angle HBA, and so the two angles HBA and HPB are equal, so the two sides HB and HP [of triangle HBP] are equal. Likewise [in triangle TGR, side] TR = [side] TG. But angle HPB is acute, since it is equal to the angle of reflection [IBZ_2, so adjacent] angle HPA will be obtuse, and HA > HP [by Euclid, I.19], so it will be greater than HB [since HB = HP]. So too, TA > TG.

[4.30] Moreover, since HP is parallel to IB, then IA:AH = AB:AP [by Euclid, VI.2]; likewise, IA:AT = GA:AR, and [conversely] AH:AI = AP:AB. But IA:AT = AB:AR, since AB = AG. Therefore, from the first, AH:AT = AP: AR [by Euclid, V.22].[17]

[4.31] But since angle HPA [in triangle HPA] is obtuse, HA^2 will exceed (HP^2 + AP^2) by twice the rectangle formed by AP and the line [segment] extended from point P to the perpendicular dropped [to AP] from point H [by Euclid, II.12]. But the perpendicular dropped from point H [to AP] will fall to the midpoint of line PB, since HB and HP are equal [as established earlier], and so HA^2 will exceed ($HP^2 + AP^2$) by AP,PB [i.e., $HA^2 - HP^2 - AP^2$ = AP,PB, so $HA^2 - HP^2 = AP^2$ + AP,PB]. Accordingly, AH^2 exceeds HP^2 by AB,AP [i.e., $AH^2 - HP^2$ = AB,AP] because (AP,PB + AP^2) = AB,AP. Likewise, AT^2 exceeds TR^2 by AG,AR, or AB,AR, which is identical [since AG = AB, both being radii of the circle].

[4.32] Accordingly, combine line AB with the two lines AP and AR [to form rectangles AB,AP and AB,AR], and the two remainders will be formed. Hence, remainder [AB,AP] is to remainder [AB,AR] as AP:AR, so ($AH^2 - HP^2$ [which = AB,AP]):($AT^2 - TR^2$ [which = AB,AR]) = AH:AT [since AB,AP:AB,AR = AP:AR = AH:AT, by previous conclusions]. And since HP = HB, and TR = TG, ($AH^2 - HB^2$):($AT^2 - TG^2$) = AH,AT.

[4.33] But [let us cut line AH at point U such that] AH,HU = HB^2. [However, AH^2 = AH,AU + AH,HU (by Euclid, II.2), so AH^2 – AH,HU (which = HB^2) = AH,AU]. Therefore, AH,AU = $AH^2 - HB^2$. So AH:AT = AH,AU:($AT^2 - TG^2$) [since AH:AT = ($AH^2 - HB^2$):($AT^2 - TG^2$), by previous conclusions, and ($AH^2 - HB^2$) = AH,AU]. And if the two lines AH and AT are combined with AU [to form rectangles, then] AH:AT = AH,AU: AT,AU. Hence, AT,AU = $AT^2 - TG^2$. So AH,HU = HB^2 [by construction], and AT,TU = TG^2 [since AT^2 = AT,AU + AT,TU, and, consequently, AT,AU = AT^2 – AT,TU. But because AT,AU = $AT^2 - TG^2$, then AT,TU = TG^2].

[4.34] Now bisect arc BG at point O [see inset to figure 6.4.3a, p. 100], drop the three perpendiculars BF, OY, and GK to line HA, draw line GS from point G parallel to HA, and drop perpendicular BX from point B to AG. If BX were extended to the circle [DGB], line AG would bisect it, as well as the arc whose chord it would form [when extended]. Accordingly, it[s other half after extension] would cut off another arc equal to arc BG, since its other arc would subtend angle GBX, and so angle GBX is half the angle [GAB] at the center [of the mirror] subtended by that same arc, according to Euclid [III.20]. Hence, angle GBX is one-half angle BAG, which line OA bisects. Therefore, angle GBX = angle OAG. Moreover, the two angles BSG and BXG are right [by construction].

[4.35] If a circle is imagined on BG [as diameter, and if it] passes through S, it will pass through X [by Euclid, III.31], and arc SX will be formed such

that the two angles XBS and XGS [i.e., AGS, since X lies on AG] will fall upon it. These two angles will therefore be equal [by Euclid, III.27]. But angle GAY = [alternate] angle XGS because of the parallelism of lines [AY and GS], so angle GAY = angle XBS. And, as has [already] been claimed, angle GBX = angle OAG. Angle OAY = angle GBS, and [so] triangle OAY will be similar to triangle GBS.[18] Hence, GB:BS = OA:AY.

[4.36] Now since angle AHB [in triangle AHB] is acute, AB^2 is less than $AH^2 + HB^2$ by 2AH,HF [i.e., $AH^2 + HB^2 - AB^2$ = 2AH,HF], according to what Euclid claims [in II.13]. Therefore, [since] DA = AB [because they are radii of circle DGB], $AH^2 + HB^2$ is greater than DA^2 by 2AH,HF [i.e., $AH^2 + HB^2 - DA^2$ (i.e., AB^2) = 2AH,HF], and thus by 2AH,HD + 2 AH,DF [i.e., $AH^2 + HB^2 - DA^2$ = 2AH,HD + 2 AH,DF, or $AH^2 + HB^2 = DA^2$ + 2AH,HD + 2AH,DF]. But 2AH,HD + $AD^2 = AH^2 + HD^2$ [by Euclid, II.7]. So, if the common [term] (AB^2 [+ 2AH,HD]) is subtracted, it will follow that HD^2 + 2AH,FD = HB^2.[19]

[4.37] But AH,HT = HD^2 [by construction], and AH,HU = HB^2 [by construction]. Hence, AH,HU = AH,HT + 2AH,DF. Having subtracted AH,HT, which we designate as common to both rectangles, it will follow that AH,TU = 2AH,DF. So TU = 2DF.

[4.38] Moreover, since angle ATG is acute, then according to previous reasoning [based on Euclid, II.13], $AT^2 + TG^2 = AD^2$ [which = AG^2] + 2AT,TK, and so [$AT^2 + TG^2 = AD^2$] + 2AT,TD + 2AT,DK. And it is proven by previous reasoning that $TG^2 = TD^2$ + 2AT,DK.[20] But AT,TU = TG^2 [by previous conclusions], and so AT,TU = TD^2 + 2AT,DK.

[4.39] Let [point E′ be chosen on line AT such that] AT,TE′ = TD^2. It follows, then, that AT,E′U [which = AT,TU − AT,TE′] = 2AT,DK [which = TD^2 + 2AT,DK − AT,TE′ (which = TD^2)], if the common term AT,TE′ is subtracted. Thus, E′U = 2DK. But it has already been claimed that TU = 2DF. It follows, then, that TE′ [which = TU − E′U] = 2FK [which = 2DF − 2DK].

[4.40] Furthermore [by Euclid, V, definition 9], AH:HT = AH:HD is a duplicate ratio [i.e., AH:HD = HD:HT], since HD is the mean proportional between them [i.e., AH and HT] because HD^2 = AH,HT [by construction]. Likewise, AT:TE′ = AT:TD is a duplicate ratio [i.e., AT:TD = TD:TE′ because AT,TE′ = TD^2 by construction]. But AT:TE′ > AH:HD.[21] Therefore, AT: TE′ > AH:HT, and since AH > AT, HT > TE′. But TE′ = 2FK [by previous conclusions].

[4.41] Moreover, as has been claimed [earlier], BG:GS = OA:OY. [So] BG:OA = GS:OY. But OA = BA [since they are radii of circle DGB], and GS = FK, according to the parallelism [of lines GS and FK and lines GK and FS]. Hence, BG:BA = FK:OY.

[4.42] In addition, IH < one-half CH [by previous conclusions], and CH = 3HT [by construction]. [So] IH < one-and-one-half HT. But HT < one-fifth HD [by previous conclusions]. Accordingly, IH is smaller yet than TD, so IH is even smaller than ND, and so MI < ND [since MI < IH]. From this it is evident that I will lie between H and Z.

[4.43] Furthermore, EZ,ZD is not greater than AD^2 [by construction, with EH as the diameter of circle QHZ], so EM,MD < AD^2. But since MG is tangent [to circle DGB], EM,MD = MG^2, according to what Euclid claims [in III.36]. Thus, MG < AD, so MG < AG.

[4.44] Also, the two triangles AGM and MGK have a common angle [i.e., AMG], and both of them have a right angle [i.e., MGA and MKG]. Hence, they are similar [by Euclid, VI.4], so MK:KG = MG:GA, and so MK < KG [since MG < GA, by previous conclusions]. And since OY > GK, HD < OY [since HD < GK < OY].

[4.45] Moreover, AH:HD = HD:HT [by construction], so AH:HD = one-half HD:one-half HT. Hence, AH:HD = QH:one-half HT, since QH = one-half HD [by construction], so AH:QH = HD:one-half HT, and so QH:AH = one-half HT:HD. But one-half HT > FK [since HT > 2FK by previous conclusions], and HD < OY. Accordingly, one-half HT:HD > FK:OY, so QH:AH > FK:OY.

[4.46] Furthermore, line AQ cuts circle EBD [represented in figure 6.4.3, p. 99]. Let Q′ [in figure 6.4.3a, p. 100] be the point of intersection, and draw line DQ′, which will be parallel to [line] QH. So QH:HA = Q′D:DA [in similar triangles AQH and AQ′D], and so Q′D:DA > FK:OY. But FK:OY = GB:BA [by previous conclusions]. Therefore, Q′D:DA > BG:BA, so Q′D > BG [since DA = BA], and arc Q′D > arc GB.[22]

[4.47] Extend AQ to point S so that AS = AI, and draw line SI, which will be parallel to [line] QH, and [so] SI:QH = IA:AH. But it was posited earlier that IA:AH = TQ:QH, so SI = TQ.

[4.48] Now because of the lack of letters for designating the key points, let us revise the diagram to avoid the excessive tangle of lines. Accordingly, since IA = the line we have designated as AS, construct circle [NRZ in figure 6.4.3b, p.101] according to their length [as radius]. Let us replace S with the letter N, let AG and AB be extended to [points R and C on the circumference of] this circle, let [the resulting lines] be ABC and AGR, and let us replace the letter Q′ with F. It has been claimed [earlier] that [arc] DF [formerly DQ′] > arc BG. Let arc BM = arc DF, and draw line AMU, as well as lines IM, NM, and [draw] line QM, and extend it to the outer circle. Let it fall to point Z, and draw lines ZA and ZG.

[4.49] Since arc BM = arc DF, then if common arc [DM] is added [to both], arc MF = arc DB. [Accordingly] angle NAM = angle IAB, the [corresponding] sides [NA and IA, and MA and BA, of triangles NAM and IAB]

will be equal [and will contain equal angles, so the triangles will be equal, by Euclid, I.4], and [so] MN = IB. And since AQ was assumed earlier to be equal to AH [because they are radii of the same circle passing through Q and H, sides] AQ and AM [of triangle AQM] = [corresponding sides] HA and AB [of triangle HAB], and angle [QAM contained by sides AQ and AM in triangle AQM is equal] to angle [HAB contained by equal corresponding sides HA and AB in triangle HAB, so the two triangles are equal, by Euclid, I.4]. [Hence] QM = HB, and angle QMN = angle HBI, since both the sides containing them [i.e., QM and NM, and HB and IB, respectively] are equal [as concluded earlier]. [Accordingly] base IH = base NQ, and angle NMU = angle IBC.

[4.50] But angle IBC [which is IBZ_2 in figure 6.4.3a] = angle HBA [by previous conclusions], and angle HBA = angle QMA, [so] angle NMU = angle QMA. And since QMZ is a straight line, as we stipulated, angle QMA = [vertical] angle UMZ, so [angle of incidence NMU = angle of reflection UMZ, and so the form of] point N is reflected to [point] Z from point M, and its image-location is [point] Q. This still falls short of a proof to show that all of [line of reflection] MZ lies outside the circle, which will be demonstrated as follows.

[4.51] It is clear that the tangent drawn from point B will fall between I and H [since HBA and IBC are acute, by previous conclusions], and [it is clear] that point B lies as far from point H as point M lies from point Q, and IH = NQ [from previous conclusions]. Therefore, the tangent drawn from point M will fall between N and Q. QM therefore intersects the circle, so all of MZ [lies] outside the circle, and so what we set out [to be proven is demonstrated].[23]

[4.52] Furthermore, because angle NMU = angle UMZ, arc NU = arc UZ. [So] angle NAU = angle UAZ. But it has already been shown that angle NAU [which is the same as angle NAM] = angle IAC [which is the same as angle IAB]. [Hence] angle IAC = angle ZAU.

[4.53] Angle BAG will be equal to, less than, or greater than angle GAM.[24] Let it be equal [as represented in figure 6.4.3c, p. 102]. Accordingly, if angle [C]BAG[R] is subtracted from angle IAB[C], and angle [U]MAG[R] from angle ZAU, angle IAG[R] will be left equal to angle ZAG[R]. [Thus], IG = ZG, triangle [IAG equals] triangle [ZAG], and angle IGA = angle ZGA. It will follow that angle IGR = angle ZGR. But angle IGR = angle TGA [from previous conclusions]. [So] angle TGA = angle ZGR. Hence, if TG is extended, it will reach [point] Z [on the circle passing through points I and N in figure 6.4.3c], so TGZ is a straight line. [The form of point] I is therefore reflected to [point] Z from point G, and [point] T is its image-location.

[4.54] So let Z be the center of sight. [The forms of] the two points N and I will be reflected to it from the two points M and G, and the [respective] image-locations will be points T and Q. Thus, TQ will be the image of line IN, and it was proven earlier that TQ = IN [i.e., IS in figure 6.4.3a], and so it can happen in these kinds of mirrors that the image is the same size as the visible object.[25]

[4.55] If, however, angle BAG > angle GAM [in figure 6.4.3d, p. 103], then angle ZAG > angle IAG. Let angle KAG = angle IAG. Since point K is lower than [i.e., to the left of] point Z, and point M is lower than [i.e., to the left of point] G, line KG will intersect line ZM. Let it intersect at point L. Accordingly, if the center of sight lies at point L, [the form of point] N is reflected to it from point M, and [point] Q is its image location; [the form of point] I is reflected to it from point G, and [point] T is its image-location, according to the preceding proof. And so TQ is the image of IN, which is what was set out [to be proven].[26]

[4.56] On the other hand, if angle BAG < angle GAM [as in figure 6.4.3e, p. 104], angle ZAG < angle IAG. Let angle OAG = angle IAG, and extend line OG. It is clear that [the form of point] I is reflected to [point] O from point G. Line OG will either intersect line ZMQ outside the [great] circle [FDGB] of the mirror, or it will not.[27]

[4.57] If it intersects outside, and if the center of sight lies at the point of intersection, [the forms of] the two points I and N are reflected to it, and T and Q are the image-locations [for those points], and so what was proposed is arrived at.[28]

[4.58] If line OG should happen to intersect line ZMQ inside the circle [as represented in figure 6.4.3e, p. 104, where X is the intersection-point], the foregoing proof cannot be applied. Instead, I say that a point can be found outside that entire surface to which [the forms of] the two points I and N are reflected from two points [on the mirror], and [that] TQ [forms] the image.

[4.59] For instance, from the foregoing it is evident that angle NAZ is twice angle IAB [since angle IAB = angle NAM = angle MAZ, by construction], and angle IAO is twice angle IAG [since angle IAG = angle OAG, by construction]. Furthermore, angle NAZ exceeds angle IAO by an amount no greater than angle NAI.[29] In addition, the two angles OAI and ZAN [together] are greater than the third [angle], which is IAN, the two [angles] OAI and IAN are greater than the third [angle] NAZ, and the two [angles] ZAN and NAI are greater than the third [angle] IAO. We therefore have three angles [IAO, IAN, and NAZ], any two of which [together] are greater than the third.

[4.60] From these [angles], therefore, a solid angle can be formed [by Euclid, XI.23].[30] Form that angle at A [in figure 6.4.3f, p. 105], let line SA be

erected at A, let angle IAS = angle IAO, [and let] angle NAS = angle NAZ. Angle NAI will remain where it is, and line AS will be formed equal to lines AN and AI, which are both equal.[31]

[4.61] Then draw lines TS and QS. It is evident that angle TAS = angle TAO [by construction], and the two [corresponding] sides [TA and OA are equal] to the two [corresponding] sides [TA and AS]. [Hence] base TS = base TO, and triangle [TAS] = triangle [TAO], and so angle [O]GTA = angle STA [since G lies on TO]. Likewise, angle QAS = angle QAZ, and the [corresponding] sides [AS and AQ are equal] to the [corresponding] sides [AZ and AQ]. [So] triangle [QAS] = triangle [QAZ], and angle [Z]MQA = angle SQA [since M lies on line QZ].[32]

[4.62] Bisect angle TAS with line AY, and let Y be the point at which that line will intersect line TS. Since angle IAG is one-half angle IAO, it is evident that angle TAG = angle TAY, whereas angle GTA = angle YTA, and one side, i.e., TA, is common [to both triangles TAG and TAY. Accordingly] TG = TY, triangle [TAG] = triangle [TAY], AY = AG, and so Y will lie on the surface of the sphere [from which the mirror is formed]. Thus, angle IAG = angle IAY, and the [corresponding] sides [IA and AG] = the [corresponding] sides [IA and AY. So] triangle [IAG] = triangle [IAY], [angle] AGI = angle AYI [and] line IY in its full extent = [line] IG.[33]

[4.63] Extend AY beyond the sphere to point P [in figures 6.4.3f and 6.4.3h, pp. 105 and 107]. Angle IGR will be left equal to angle IYP. But, since TS = TO, and TY = TG, it follows that GO = YS. Therefore, AY and YS are equal to AG and GO [respectively], and base AS = base AO. [Hence] triangle [AYS] = triangle [AGO], and [so] angle AYS = angle AGO. It follows that angle SYP = angle OGR. Thus, the two angles IGR and OGR are equal to the two angles IYP and SYP [respectively].

[4.64] But line AS intersects the sphere [of the mirror]. Let O′ [in figure 6.4.3h] be the point of intersection. Accordingly, the three points O′, Y, and D lie on the surface of the sphere, so line O′YD is a segment of a [great] circle of the sphere, and it is the common section of the sphere's surface and plane ITASP, so [the form of] point I is reflected to point S from point Y [within plane of reflection ITASP], and T is the image-location.

[4.65] Likewise, if angle NAS [in figure 6.4.3f. p. 105] is bisected by [line] AZ′Z″, it will be proven in the preceding way that QZ′ = QM, AZ′ = AM, Z′S = MZ, and the two angles NZ′Z″ and SZ′Z″ are equal to the two angles NMU and ZMU. And so [the form of point] N is reflected to [point] S from point Z′, and Q is the image-location, so TQ is the image of IN, which is what was set out [to be proven].

[4.66] Now if a perpendicular is dropped from point I to NA, it should fall between N and Q, not beyond N because angle INA is acute, since it is

equal to angle NIA, and if that perpendicular were to fall beyond N, an acute [angle] would be greater than a right angle.[34] Therefore, that perpendicular will form a right angle on NQ, and that angle will be subtended by line IN, so line IN > that perpendicular, and so that perpendicular < TQ [since TQ is equal to IN by the initial construction].

[4.67] The [form of the] point on line NQ where the perpendicular falls is reflected to point S, and its image will lie on line NA above point Q because the farther away [from the mirror's surface] the points that are reflected lie, the more their image-locations approach the center of the circle, according to the tenth [proposition] of the fifth book.[35]

[4.68] Moreover, any line drawn from point T to any point on NQ above Q will be longer than TQ. Therefore, the image of the perpendicular will be longer than the perpendicular itself [which is shorter than IN]. By the same token, no matter what line is drawn to NQ from point I between this perpendicular and IN, its image will be longer than it.

[4.69] But these claims may be determined more definitively [as follows. The form of] point N [in figure 6.4.3m, p. 108] is reflected to [point] Z from point M, and Q is the image-location. Line QM cuts the circle at point E. Therefore, the tangent drawn from point Z to the circle will fall at some point on arc ME, and that tangent will fall above Q, since the point where it will fall [on NQ] will form the endpoint of tangency [X on cathetus NA] and [thus] the limit of images, and points below that endpoint of tangency cannot be reflected, [whereas points] above it can.[36]

[4.70] Therefore, if the perpendicular dropped from point I falls above the endpoint [X] of tangency, the point where it falls is reflected, and the image of the perpendicular will be longer than the perpendicular [itself]. But if the perpendicular should fall at or below the endpoint of tangency, the point where it falls is not reflected, so there will be no image of the perpendicular. However, since the endpoint of tangency lies below N, there will be an infinitude of points between the endpoint of tangency and N, and any of them will be reflected, and the image of any of them [will lie] on NQ. And the image of any line drawn from point I to any of those points will be longer than the line of which it is the image.

[4.71] In these [sorts of] mirrors, then, the image may sometimes be the same size as the visible object and sometimes larger, which is what was set out to be explained. Moreover, we have not read an explanation of this matter in any text, nor have we heard anyone who has discussed it or thought about it.

[4.72] Moreover, in these [sorts of] mirrors straight lines appear curved, such that, in many cases, the curvature [of their images] does not correspond

to that of the mirror but is opposite. Likewise, curved [things] will appear curved in these [sorts of] mirrors, and if the curvature corresponds to that of the mirror, it will appear in an opposite orientation, but this must be understood not [to hold] in all cases but in several, [but] for the sake of explaining this, certain preliminary points must be set out, one of them being as follows.

[4.73] **[PROPOSITION 4, LEMMA 1]** If two points lie the same distance from the center of the mirror and different distances from the center of sight, the image of the point lying farther from the center of sight will lie farther from the center of the sphere [forming the mirror] than the [image of the point] lying nearer [the center of sight], and the endpoint of tangency for the farther [point will lie] farther from the center [of the circle] than the endpoint of tangency for the nearer [point, and this will be the case] whether those points lie in the same plane as the center of sight or in different planes.

[4.74] The proof [is as follows]. Let T and D [in figure 6.4.4, p. 109] be two points equidistant from G, the center of the mirror, [and let] E be the center of sight. Plane DGT will cut the mirror along [great] circle AB. Let angle EGD = angle TGZ, [let] angle EGT = angle TGH, and find point Q on the circle from which [the form of point] T is reflected to [point] Z [by book 5, proposition 25, in Smith, *Alhacen on the Principles*, 427-432].[37] I say that [the form of point] T is not reflected to [point] H from any point on BQ.

[4.75] It is obvious that [it does] not [do so] from point Q [itself]. Moreover, if some point is taken on BQ, the line [of reflection] drawn to that point from point H will intersect line QZ. [The form of point] T is therefore reflected to that point of intersection from the point selected on BQ, and it is [also] reflected to that same point of intersection from point Q. So [the form of] point T is reflected to the same point from two points on that circle, which is impossible in these [sorts of] mirrors, as was shown in [proposition 16 of] the fifth book [in Smith, *Alhacen on the Principles*, 412-414].

[4.76] It follows that [the form of point] T may be reflected to H from some point on QA. Let that [point] be M [as found by book 5, prop. 25], and from point M draw MN to line GT tangent to that circle. N will be the endpoint of tangency for T with respect to H [as center of sight].[38]

[4.77] Then from point Q draw tangent QO, which will necessarily lie below MN. Extend ZQ until it falls on GT at point C. C will be the image-location [of T] for Z [as center of sight]. Thus, GT:TO = GC:CO [by book 5, proposition 7, in Smith, *Alhacen on the Principles*, 404]. So GT:TN > GT:TO. *A fortiori*, then, GT:TN > GC:CN. Accordingly, let GT:TN = GL:LN.

GL > GC, and L will be the image-location [of T] for [center of sight] H [according to book 5, prop. 7].

[4.78] So let lines HG, EG, and ZG be equal, [let] GF = GC, [and let] GS = GO. Therefore, since angle EGD = angle TGZ [by construction], and since D lies as far from point E as Z does from point T [given that DG = TG, and EG = ZG, by construction], the image of D with respect to G will lie as high on line GD as the image of T on line GT [with respect to G, by book 5, proposition 17, in Smith, *Alhacen on the Principles*, 414-415]. Thus, the image of [point] D [with respect to G lies] at point F [since GF = GC, by construction]. Likewise, the endpoint of tangency for D with respect to E will lie at the same height as the endpoint of tangency [at point O] for Z, so the endpoint of tangency for D [lies] at point S [since GS = GO, by construction].

[4.79] But since angle EGT = angle TGH [by construction], and since HG = EG [also by construction], L will be the image of T with respect to E, just as it is with respect to H. And N is the endpoint of tangency with respect to E, so the image [L] of the point farther from E [i.e., T] lies farther from the center than the image [F] of the nearer [point D], and the endpoint of tangency [N] for the farther [point T lies] farther from the center than the endpoint of tangency [S] for the nearer [point D], which was what was set out [to be proven].

[4.80] **[PROPOSITION 5, LEMMA 2]** Furthermore, given line AB [in figure 6.4.5, p. 109] divided at points G and D such that AB:BD = AG:GD, I say that, if three lines, i.e., GE, DE, and BE, are drawn from the points of division to intersect at one point, and if a line is drawn from point A to intersect those three lines, that line will be cut according to the aforesaid proportion.

[4.81] The proof [is as follows]. Draw line AT to cut the three sides GE, DE, and BE at the three points Z, H, and T. I say that AT:TH = AZ:ZH.

[4.82] From point H draw HQ parallel to AB. It is clear that AB:BD is compounded from AB:HQ and HQ:BD [i.e., AB:BD = (AB:HQ)(HQ:BD)]. But since QH is parallel to AB, triangle TQH will be similar to triangle BTA, and [so] AB:QH = AT:TH. Likewise, triangle QEH is similar to triangle BED. Therefore, QH:BD = HE:ED. Hence, AB:BD is compounded of AT: TH [which = AB:QH] and HE:ED [which = QH:BD].

[4.83] Extend QH until it falls on EG at point M. AG:GD is compounded from AG:HM and HM:GD. But since angle EMH = [alternate] angle ZGD, then angle HMZ [adjacent to EMH] = angle ZGA [adjacent to ZGD], and [so] triangle AZG will be similar to triangle HZM, and AZ:ZH = AG:HM.

[4.84] But triangle HEM is similar to triangle GED. [Hence] HM:DG = HE:ED. Accordingly, AG:GD is compounded from AZ:ZH and HE:ED, and

AG:GD = AB:BD [by construction].[39] Thus, that same [proportion AG:GD] is compounded from AT:TH and HE:ED, and it is likewise compounded from AZ:ZH and HE:ED. Hence [if we drop the common term HE:ED], AT: TH = AZ:ZH, and so what was set out [has been proven].

[4.85] The same proof will hold no matter what line is drawn from point A to intersect those three intersecting lines. And if three other lines are drawn from the three point G, D, and B to intersect at some point other than E, and if any line is drawn from A to intersect those [three lines], it will be cut according to the aforesaid ratio. And so, however the three lines may intersect, if the [resulting] three lines [represented by] EG, ED, and EB are extended beyond the three points B, D, and G, on the other side [away from the point of intersection], and if lines are drawn from point A to intersect them on that other side, those lines may never be cut according to the aforesaid ratio.[40]

[4.86] **[PROPOSITION 6, LEMMA 3]** Moreover, given line AB [in figure 6.4.6, p. 110] cut in the preceding way, if another line, such as AT, is drawn from point A so as to be cut according to the same ratio, and if lines are drawn from the points of division on AB to the points of division on AT, those lines not being parallel, I say that those three [lines] will intersect at the same point.

[4.87] The proof [is as follows]. Let AT:TH = AZ:ZH. BT and DH are not parallel, so they will intersect at point E. Line GZ will either intersect [them] at the same point, or it will not. If [it does intersect] at that point, we have what was set out [to be proven]. If not, then draw line EG. It will intersect line AT in some point other than Z. Let that point be L. Thus, AT: TH = AL:LH, according to the previous proof. But it has been supposed that AT:TH = AZ:ZH, and so it is impossible [for EG to intersect AT at some point other than Z].

[4.88] Likewise, if it is supposed that line GZ intersects DH at point E, it will be proven in this way that line BT will intersect [it] at the same [point]. So too, if it is supposed that GZ and BT intersect at point E, it will be indisputable that DH will intersect at the same [point].

[4.89] **[PROPOSITION 7, LEMMA 4]** Furthermore, given that AB [in figure 6.4.7,p. 110] is divided according to this ratio [AB:BD = AG:GD], if lines GZ, DH, and BT are parallel, and if AT is drawn to cut them, AT will be divided according to this ratio.

[4.90] The proof [is as follows]. Since DH is parallel to GZ, AZ:ZH = AG:GD, and since BT is parallel to DH, AB:BD = AT:TH. But AB:BD = AG: GD, [so] AT:TH = AZ:ZH, and so [we have demonstrated] what was set out

[to be proven]. With these points established, let us proceed to what was proposed [in paragraph 4.72, p. 175-176 above].

[4.91] **[PROPOSITION 8]** First of all, it must be shown how in these [sorts of] mirrors the image of an arc is curved with a curvature that accords not with the [surface of the] mirror but with its center.

[4.92] For instance, let AB [in figure 6.4.8, p. 110] be an arc facing the mirror [composed from sphere YZ on whose surface Y′Z′ is a great circle within the plane of arc AB], let G be the center of that arc as well as of the mirror, [and let] D [be] the center of sight. Draw lines DG, AG, and BG. Take E at random on arc AB, and draw line EG. Let line DG not lie in plane ABG. Line DG will either be perpendicular to plane ABG, or [it will be] inclined [to it].

[4.93] Let it be perpendicular. Angles DGA, DGE, and DGB will be equal, and the [corresponding] sides [of triangles DGA, DGE, and DGB will be equal] to the [corresponding] sides, so the bases [DA, DE, and DB will be] equal. Hence, all the points on arc AB will lie the same distance from the center of sight, so the images of all of them [will lie] the same distance from the center.

[4.94] Let Q, M, and L be the images of A, E, and B. Accordingly, GQ will be equal to GM and GL, so QML will be an arc, and its convex curvature accords with the center [of curvature], not with the [surface of the] mirror, or with the points of reflection, which is what was set out [to be proven].[41]

[4.95] If, however, line DG is not perpendicular to plane AGB [as in figure 6.4.8a, p. 111], and if a perpendicular [DX] is dropped from point D to that plane, then, since that perpendicular is the shortest of all [possible] lines extending from point D to this plane, the angle this perpendicular forms with respect to G will be smaller than any angle imagined at point G formed by any other line drawn from point D to the plane, and the farther the line drawn from point D to the plane will lie from the perpendicular, the longer it will be and the greater the angle it will form. Accordingly, if this perpendicular does not fall on arc AEB but on one side of it, all the lines drawn from point D to this arc will be slanted to one side, and the ones that lie farther away will be longer and will form a larger angle.[42]

[4.96] Let [this] be [the case], then, and take three points, i.e., E, C, and B, on the arc [in figure 6.4.8b, p. 111].[43] Let L be the endpoint of tangency for point B, and let M be the endpoint of tangency for point C, for, since C is nearer than B to D, M will be nearer than L to G, and so CM > BL.[44]

[4.97] Let Q be C′s image, let T be B′s image, and draw TQ. Then draw lines CB and ML, which will intersect when extended, for if a line were

drawn from M parallel to CB, it would cut a line from GB equal to CM [and thus be shorter than GL, which means that angle MLG < angle CBG]. Let them intersect at point O.

[4.98] Since GC:CM = GQ:QM [by book 5, prop. 7], and likewise, since BG:BL = GT:TL [by the same proposition], line QT will intersect lines CB and ML [all at the same point]. Let the intersection be at point O.[45]

[4.99] Let N [in figure 6.4.8b, p. 111] be the endpoint of tangency for point E. Since point N is lower than point M [by proposition 4, lemma 1], EN > CM. Hence, if they are extended, lines EC and NM will intersect. Let the intersection be at point P, draw line QP, and extend it until it falls upon EG at point F. Then extend line [O]TQ to EG, and let it fall at point K.

[4.100] It is evident that K will lie above F. However, since GC:CM = GQ:QM, and since three lines [EP, NP, and FP] are drawn through the points of division to meet [at point P] when extended to the other side, they will cut line EG according to the previous ratio [by proposition 6, lemma 3], so GE:EN = GF:FN. But N is the endpoint of tangency [for E, by construction], so F is the image-location [for E]. Hence, line FQT will be the image of arc ECB, and it will be a curved line rather than a straight one, for TQK is straight [by construction], and the curvature of the [resulting] line [TQF] is not oriented with [that of] the mirror.

[4.101] Likewise, if the perpendicular dropped from point D falls on the other side of the arc [i.e., to the left of perpendicular D'G], the proof will be identical. But if the perpendicular falls on the midpoint of arc AB, the lines drawn to the arc from point D to opposite sides and equidistant from the perpendicular will be equal and will form equal angles at G. Moreover, their images will be equidistant from G, and so will the endpoint[s] of tangency,[46] and it can be proven in the foregoing way for either side of the arc by itself, according to how it is cut by the perpendicular, that its image is a curved line in the way prescribed, which is what was set out [to be proven].

[4.102] **[PROPOSITION 9]** Now take a circle [containing arc ADB in figure 6.4.9, p. 112] whose center [F] is not the center [G] of the mirror [containing arc HK]; nonetheless, let it lie in the same plane as the mirror's center. I say that, if an arc [ADB] is taken on the [larger] outer circle on the side of the mirror's center, i.e., nearer that center [G and thus directly opposite it], its image will be curved.

[4.103] For, given this arc, draw a line [FG] from the center of the mirror to the center of the outer circle, and extend this line to the given arc [ADB]. The line drawn from the center of the mirror to this arc [i.e., GD], which is a segment of the diameter [FD] of the larger circle, will be shorter than all the [other] lines drawn from the same centerpoint of the mirror to that arc.

Moreover, two equal lines [GA and GB] can be drawn to the given arc from the mirror's centerpoint on opposite sides of this shortest line, and they will of course be longer [than it]. And if a circle is drawn according [to the length of] either of them from the center of the mirror, arc [AEB on it] will pass through the endpoints of these two lines, and it will be longer [and of a sharper curvature] than the given arc [ADB].

[4.104] It is clear that the image of this longer arc will be a curved line, according to previous conclusions [in proposition 8]. [It is] also [clear that] the images of the points [A and B] common to this arc [AEB] and the given arc [ADB will be] the same and [that] the midpoint [E] of the longer arc lies farther from centerpoint [G] than the [mid]point [D] on the given arc that corresponds to it, so its image [will lie] nearer to the centerpoint [G] than the image of point [D] on the given arc that corresponds to it [by book 5, prop. 17, in Smith, *Alhacen on the Principles*, 414-415]. And so, the image of any point on the outer arc [will lie] nearer the centerpoint than the image of the point on the given arc that corresponds to it. Accordingly, the image of the given [less sharply curved] arc [ADB] is more sharply curved than the image of the [more sharply curved] outer arc [AEB], so the image of the given arc is curved, which is what was set out [to be proven].

[4.105] **[PROPOSITION 10]** In addition, it is proven as follows that the image of a straight line is curved in these [sorts of] mirrors.

[4.106] Let A[C]B [in figure 6.4.10, p. 112] be the [straight] line that is seen, [and let] G [be] the center of the mirror. Draw lines AG and BG. They are either equal or not. If [they are] equal, then construct [the] circle [containing arc] AEB on centerpoint G according to their length. Line A[C]B will obviously fall inside [this] circle. From the previous [proposition] it is clear that the image of arc AEB will be curved. Accordingly, let its image be ZTH. Let Z be the image of A, let H be the image of B, and let T be the image of E.

[4.107] Extend line GE to cut AB at point C. It is clear that E lies on the same line with C [and] farther from the centerpoint [G] than C. Its image will [therefore] lie nearer to centerpoint [G] than the image of C [by book 5, prop. 17]. So let it be M. It is clear that line ZMH is the image of line AB, and it is a curved line, which is what was set out [to be proven].

[4.108] **[PROPOSITION 11]** On the other hand, if lines AG and BG are not equal, then, when it is extended, line AB will either intersect the mirror or not. Let it not intersect [as in figure 6.4.11, p. 113], let AG > BG, construct circle AEQ at [centerpoint] G according to the length of AG [as radius], and extend AB until it touches the circle on the side of B. Let it fall at point Q.

[4.109] It is clear from the foregoing [analyses in propositions 8 and 9] that the image of arc AE is curved. Let Z be the image-point for A, and let M be the image-point for E. ZM will [therefore] be the image of arc AE, and since the image of point B lies farther from centerpoint [G] than the image of point E [by book 5, prop. 17], the image [TNZ] of line AB will be curved, [and] this can be demonstrated according to the midpoints of arc AE and line AB, which is what was set out [to be proven].

[4.110] Note that in the preceding figure, if a segment is cut from line AB on the side of A, and a segment is cut on the side of B equal to it, the remaining portion of the line will have a curved image, and the proof [of this] will be the same as it is for [the whole of] line AB. Also, if in this figure another segment of line AB is cut on the side of B, the same proof will hold for the remainder [of the line] as holds for [the whole of] line AB.

[4.111] **[PROPOSITION 12]** But if line AB touches the mirror, it will either intersect it or be tangent to it. Let it be tangent [as in figure 6.4.12, p. 114]. Let G be the center of the mirror [containing arc PE in gray], and draw lines AG and BG. Plane ABG cuts the mirror along the circle SEZ [that forms their] common [section]. It is clear that line AB will be tangent to the mirror on this circle. Let it be tangent at point E. Accordingly, extend AB to E. Let D be the center of sight. The plane containing lines DG and AG cuts the mirror along a [great] circle [that forms the] common [section] of the plane and the mirror. Let ZP be an arc on that circle. Likewise, let HP be an arc on the [great] circle [that forms] a common section of the plane containing DG and BG [and the mirror].[47]

[4.112] It is clear [from proposition 4, lemma 1, paragraphs 4.74-76] that [the form of point] B is reflected to [point] D from some point [F′] on arc HP. If a tangent is drawn from that point, it will intersect line BG, and the point of intersection will be the endpoint of tangency [for point B on cathetus BG]. Let M be that point [on the resulting tangent F′M].

[4.113] It is also clear that, if a tangent is drawn from point M to circle SEH, that tangent will fall in front of E because AB is tangent at point E, and point B lies above point M. It will therefore fall at point F, and, when it is extended, the tangent [MF] will intersect line AE. Let it intersect at point T. It will intersect line AG on the other side. Let it intersect at point C.

[4.114] Form angle BGS = angle BGD, and extend GS to point L so that it is equal to line DG. Accordingly, arc HS = arc HP, and just as [the form of point] B is reflected to [point] D from a point on arc HP, it[s form] is reflected to L from some point on arc HS. Moreover, the reflection will occur from point F just as the reflection on arc HP occurs from the point [F′] from which the tangent [F′M′] is drawn to point M, and those two points

lie the same distance from point M [which means that the reflections from B to D and from B to L are perfectly equivalent]. So draw lines BF and LF.

[4.115] [The form of point] A is reflected to D from some point [R] on arc ZP. However, in triangle HZP the two arcs HZ and HP are longer than the third, i.e., ZP.[48] But HP = HS. Therefore, ZP < ZS. Cut an equal segment from ZS at point Y [so that ZY = ZP], and draw line GY, which, when it is extended to the same length as GD, will necessarily intersect line FL. Let it intersect at point X, and let GXK = GD.

[4.116] It is clear that, just as [the form of point] A is reflected to D from some point [R] on arc ZP, it is likewise reflected to K from some point [R′] on arc ZY.[49] I say that it is reflected to it [i.e., K] only from a point that is below F on the side of Z.

[4.117] For if it is claimed that it can [be reflected] from point F or from some point on arc FY, the line drawn from point A to the point of reflection will intersect line BF. [The form of] point K is reflected to that point of intersection, and [the form of] point L is reflected to the same point, and so two points are reflected in these [sorts of] mirrors to the same point on the same side, which is impossible [by book 5, prop. 16]. It follows that [the form of] point A is reflected to [point] K from some point [R′] on arc ZF.

[4.118] If a tangent is drawn from that point, it will intersect line AZ, and it will fall between C and Z because point F is lower than any [other] point on arc ZF, so the tangent from point F is higher than the rest that are drawn from points on arc ZF. So let that tangent [R′N] fall at point N, and draw line NM, and since it passes through the vertex of triangle BMT and cuts the angle when extended, that line will necessarily intersect BT. Let it intersect at point Q, and draw line GQ.

[4.119] Now let I be the image of point A; let O be the image of point B; and let U be the image of point Q. Since B lies nearer than A to point G, O will lie farther than I from point G [by book 5, prop. 17]. So draw line IO. It is also clear [from book 5, prop. 7] that AG:AN = GI:IN, while BG:BM = GO:OM. Thus, since lines AG and BG are each cut in three points according to this ratio, and since two of the lines, i.e., AB and MN, extended from the points of division [A and N on base AG] intersect at the same point, i.e., at the same point Q, the third [line extending from point I on base AG] will necessarily intersect [these two] at that same point [by proposition 6, lemma 3].

[4.120] Therefore, when it is extended, IO will fall upon Q, so IOQ [forms] a straight line. Thus, IOU will not be a straight [line]. But IOU is the image of line AQ, so the image of line AQ will be curved. Furthermore, if point B replaces point Q, and if some point on line AB replaces point B, it will be demonstrable in exactly the same way that the image of line AB is curved, and this is what was set out [to be proven].[50]

[4.121] **[PROPOSITION 13]** If, however, AB intersects the circle [in figure 6.4.13, p. 116], let it intersect at point E, [and let] M [be] the endpoint of tangency on line BG. [The form of point] B is reflected to [point] D from some point [F′] on arc HP. The arc [extending] from that point of reflection to H [i.e., HF′] is either equal to, longer than, or shorter than arc HE.

[4.122] If it is equal (but it is evident that that arc is equal to arc HQ), let Q [in figure 6.4.13] be the point on the circle where the tangent drawn from point M falls on the side of E. Thus, AE passes through point Q, so MQ intersects AE through point E.[51]

[4.123] But if that arc [HF′] < arc HE [as in figure 6.4.13b, p. 118], MQ will intersect line AE beyond point Q [at point T] so as to form triangle EQT.[52]

[4.124] On the other hand, if that arc [HF′] > arc HE, line MQ will intersect line AE below point Q.[53]

[4.125] Whether the latter or the former is the case, repeat the earlier proof, and it should be demonstrated in precisely the same way that the image of line AB is curved, which is what was set out [to be proven].

[4.126] **[PROPOSITION 14]** Furthermore, if the center of sight lies in the plane containing the visible line and the center of the sphere—in the previous cases it was stipulated that the center of sight does not lie in that plane[54]—then the straight visible line will either intersect the [great] circle [forming] the common section of that plane and the mirror, or it will not intersect [it].

[4.127] If it will intersect, it will be either perpendicular to the mirror['s surface] or inclined to it. If [it is] perpendicular, the angle [formed by] those lines will fall on the center of the mirror, and [the image of] that line will appear straight, for the image of any point on that line will appear on that line, and so the image of that line [will be] straight.

[4.128] But if the given line [AB] is slanted, its slant will either be toward the center of sight or away from it. If it slants away from it [as in figure 6.4.14, p. 120],[55] find the point [R] on the circle from which [the form of] some [point on it, such as B′] is reflected to the center of sight [D], and find the line of reflection [RD]. Any of the slanted lines may fall on this line of reflection, and if it does, then [an image of] this slanted line will not be seen.

[4.129] Having extended a line [DG] from the center of sight to the center of the mirror, take a point [R′] on the arc of the circle in front of this line, such that [the form of] some point [B′] on the slanted line is reflected from it to the center of sight. But [the form of] that point is reflected from the previously designated point [R], which is the endpoint of the line of reflection, since the slanted line lies upon the line of reflection, and so [the

form of] that point on the slanted line is reflected to the center of sight from two points on the arc, which is impossible [by book 5, prop. 16].

[4.130] Moreover, even though [the form of] that point may be reflected from the point that is initially selected, it[s image] is still not seen, since it lies on the line of reflection, so it[s image] is occluded by points in front of it [on the object-line], and so [the image of] a line lying upon the line of reflection is not seen.[56]

[4.131] But if a slanted line [AB in figure 6.4.14a, p. 120] is taken with its slant not toward the center of sight, and if it lies below the line of reflection [AB] and cuts it at a point [B] on the circle, I say that no [image of any] point on that line will be seen.

[4.132] For, given [such a] point [e.g., A], if it is claimed that [the form of] that point can be reflected from some point [R′] on the arc lying between the line of reflection [DB] and line [DG] extended from the center of sight to the center of the mirror, and if a line [of incidence AR′] is extended from that point to the point chosen on the arc, this line will intersect the line of reflection [BD], and the point of intersection [X] is reflected to the center of sight from two points on the arc, which is impossible.[57]

[4.133] On the other hand, if it is claimed that the [form of the] point [A] taken on the [slanted] line is reflected from a point on the arc of the circle below that line [i.e., to the right of B], it will be impossible [for it to be seen], since that whole arc is occluded by the line.

[4.134] If, however, the chosen line does not reach the circle, it can indeed be seen, but it is quite small. But if a line with the previous slant [i.e., away from the eye] is selected between the line of reflection and the line initially assumed to pass through the point of reflection to the center [of the circle], this line can in fact be seen, and the curvature of the image of this line will decrease as it approaches the line passing through the point of reflection to the center [of the circle].[58]

[4.135] But if lines are chosen between the line passing through the point of reflection to the center [of the circle and the mirror], they will appear, whether their slant lies toward the center of sight or not. And the way they [i.e., lines slanting toward the eye] are seen is like the way lines [slanting away from the eye] between the line of reflection and the line passing to the center are seen. But these things must be understood for lines that meet the arc of the visible part of the circle, i.e., on the arc lying between the two [lines] drawn tangent to the circle from the center of sight.

[4.136] On the other hand, among lines that meet the circle on the side of the circle that is invisible, one of them will be parallel to the line of reflection. That one will not be seen. Likewise, any one that borders on the parallel and lies below it will be invisible, whereas one that borders on the parallel [and lies] above it can be seen.[59]

[4.137] If, however, a line is selected between the parallels but not bordering on any of them, and if it is slanted toward the center of sight, it will be seen. If it slants in the other direction, it will sometimes be seen, and sometimes not because, if a parallel to the line of reflection is extended from its endpoint, and if that line lies below the parallel, then it will not be seen, whereas [if it lies] above it can be seen.[60]

[4.138] But if the lines do not meet the circle, they will either intersect the line drawn from the center of sight to the center of the mirror, or they will be parallel to it. If any of them intersects it, that line will either intersect it on the side of the center of sight, i.e., between the center of sight and the mirror, or [it will intersect it] beyond the mirror [and the center of sight]. If [it intersects] beyond [i.e., above the head], that line will not be visible, but its ends may appear. If it cuts the visual axis on the side of the center of sight, it will in fact appear the same [i.e., not visible in the mirror]. If it is parallel to the visual axis, it can be seen.[61] Moreover, the images of all these lines are curved.

[4.139] And if the center of sight lies in the same plane as the center of the mirror and the visible lines, they appear diminished, and the one that appears most clear in this case is the one that is most slanted and that corresponds to the center of sight. By the same token, the images of arcs that appear in these [sorts of] mirrors and that lie in the same plane as the center of the mirror and the center of sight appear curved according to the curvature of the mirror.[62]

[4.140] And these things must be understood [to apply] when both eyes lie in the same plane as the center of the mirror and the object that is seen. For if either eye lies slightly outside [that plane], the object will be perceived in another way. And if the eye lies outside the plane containing the visible object and the center of the mirror, it will be perceived more clearly than if the eye lies in that plane.[63]

[4.141] **[PROPOSITION 15]** That the image of a visible object is curved when the center of sight lies in the plane containing the mirror's center and the visible object will be proven [as follows].

[4.142] Let D be the center of sight and G the center of the mirror. Let HE be the visible line. Let HE not intersect the circle but be parallel to line DG [as in figure 6.4.15, p. 122], or let it intersect it on the side of D [as in figure 6.4.15a, p. 122]. Take the plane containing line DG and line HE, and let circle AB be the common section of this plane and the mirror.

[4.143] Draw line HG. Let Z be the image of H, let B be the point on the circle from which [the form of point] H is reflected to [point] D, let the tangent be drawn from point B, and let it intersect line HG at point T. T will be the endpoint of tangency [on cathetus HG].

[4.144] Draw line GB, which will necessarily intersect HE when it is extended, for, if HE is parallel to DG, it will necessarily intersect. If, however, DG intersects HE, then *a fortiori* GB will intersect it. That intersection will lie either on line HE, or beyond that line.

[4.145] Let it lie beyond. Let it intersect at point M, let Q be the image of point M, and let S be the endpoint of tangency [on cathetus MG]. Draw line ZQ, as well as line TS, and draw tangent AU from point A. It is clear that [arc] AB is less than one-fourth [the circumference of the circle, since GD lies on a diameter of the circle and DB intersects the circle], so [the eye at point] D should see less than half the circle [when the corresponding arc below A is included], [and] so angle AGB is acute, while angle UAG is right. Hence, AU will intersect GB. Let it intersect at point U. I say that point U should fall above point S.

[4.146] For, since [the form of] point M is reflected from some point [X] on arc AB, and since A lies below that point, the endpoint of tangency for A [as a point of reflection for the form of any point on cathetus GM] will lie higher than the endpoint of tangency for that point [X as a point of reflection for any point on cathetus GM]. And so S [lies] below point U. Accordingly, extend TS until it intersects line AU, and let the intersection be at point K.

[4.147] Draw line GK, and let it intersect HM at point C when it is extended. [The form of] point C is reflected to [point] D from some point on arc AB. Let F be that point, and from it draw a tangent to GC, that tangent being lower than line AK, and [any] point [on it, such as] O will be lower than point K.

[4.148] Let O be the endpoint of tangency. Extend line DF until it falls on GC. Let it fall at point R. Extend ZQ to line GC, and let it fall at point L. I say that L lies above R.

[4.149] For either lines HC, TK, and ZL are parallel, or they will intersect. Let them be parallel [as in figure 6.4.15a]. Accordingly, since they are parallel, let them intersect line CG at the three points C, K, and L, and let them intersect both lines MG and HG. HG:HT = GZ:ZT [by book 5, prop. 7], and likewise MG:MS = GQ:QS [because HG and GM are cut equiproportionally by parallels HMC, TSK, and ZQL, and for that same reason] GC:CK = LG:LK [all according to proposition 7, lemma 4].

[4.150] But it is clear that R is the image of C because line of reflection DF intersects CG at point R, and O is the endpoint of tangency, so GC:CO = GR:RO [by book 5, prop. 7]. However, GC:CK [which = GL:LK] > GC:CO [which = GR:RO], and so GL:LK > GR:RO. Accordingly, OR:RG > KL:LG, and so OG:RG > KG:LG [by Euclid, V.18]. But KG > OG, so LG > RG. Hence, R lies lower than point L. But ZQL is a straight line. Therefore, ZQR is a curved line, and so the image of line HC is curved. So if some

point on line HC replaces point M, and if point E replaces point C, it will be demonstrable that the image of HE is curved.

[4.151] But if lines HC, TS, and ZQ intersect, the intersection will either be on the side of D, or [it will be] on the side of HG. Let it be on the side of D [as represented in figure 6.4.15b, p. 123], and let the intersection be at point C. ZQC will be a straight line, so ZQR will be curved, and so the image of line HE [will be] curved, which is what was set out [to be proven].[64]

[4.152] If an arc is posed outside the mirror, however, it will be possible from this to prove that its image is curved just as it was proven [in proposition 11] when the center of sight did not lie in the same plane as the arc and the center of the mirror, and this is what was set out [to be proven].

[4.153] Therefore in these [sorts of] mirrors straight lines appear curved, and likewise curved lines appear curved. Moreover, if a curved object is placed before the eye in these mirrors, and if it is long and has some slight breadth, the curvature of that object will appear clearly, since it can be detected by those features lying on or within the body. In fact, unless it is considerable, the curvature is not clearly detected when the boundaries of the length or breadth are hidden [so that the image extends beyond the visible face of the mirror], so when an object of slight curvature and considerable size is placed before the eye, its curvature is not clearly detected, even though its image is curved, since the boundaries along the length and breadth of the object do not appear [in the mirror].

[4.154] Moreover, all of the errors that occur in plane mirrors occur in these mirrors as well,[65] and besides those [errors] it happens that the images of straight lines are curved, which is far from the case in plane mirrors.

[CHAPTER 5
On Convex Cylindrical Mirrors]

[5.1] Now the same errors occur in convex cylindrical mirrors as occur in convex spherical mirrors, for [in the former] as in the latter, straight lines appear curved and the size of the visible object appears diminished, but far more pronouncedly because in [convex] spherical mirrors a large object will appear smaller, to be sure, but not very small, whereas in convex cylindrical ones even a very large object will appear greatly diminished [in size]. Likewise, a straight line will appear curved in [convex] spherical mirrors, but if it is [even] slightly curved [it will appear] extremely so in [convex] cylindrical [mirrors], so the errors [that occur] in a [convex] spherical mirror are compounded in a [convex] cylindrical mirror.

[5.2] However, in [convex] cylindrical [mirrors] reflection sometimes occurs from a straight line, i.e., [when it occurs] from [a line of] longitude on the mirror, sometimes from a circle [on the mirror's surface], and sometimes from a [cylindric] section [i.e., an ellipse, on that surface]. When the visible line is parallel to a [line of] longitude on the mirror, the reflection will occur from [that] line of longitude, and a straight visible line will appear [only] slightly curved. These things, moreover, will be demonstrated, but for that demonstration a preliminary point must be made, as follows.

[5.3] **[PROPOSITION 16, LEMMA 5]** If a cylindric section [e.g., elliptical section APEBR in figure 6.5.16, p. 124] is assumed and some point [E] is taken on it that is not a point of reflection, then, when a line [ED] is extended from that point to the normal [BD dropped] from the point of reflection [B] to the axis [of the cylinder on which the ellipse is chosen]—and that line should form an acute angle [EDB] with the normal—if a line [EU] is drawn perpendicular to the tangent [QEL] at that point [E], this line will intersect the normal [BD] outside the axis and outside the intersection of the previous line [ED] with the normal [BD].[66]

[5.4] For example, let AEB be the [assumed cylindric] section, E the given point [that is not the point of reflection], N the visible [object-]point, B the point of reflection, BD the [given] normal [dropped from the point of reflection to the axis], EDB an acute angle, and QEL the tangent [to the cylinder as well as to the elliptical section at the chosen point E].

[5.5] At point B form a circle, i.e., BTO, parallel to the cylinder's base, and draw a line of longitude through point E on the cylinder, i.e., ET. Draw axis DH [of the cylinder], and draw line DC perpendicular to BD.

[5.6] It is obvious that plane HDC is orthogonal to the plane of the circle [BTO]. But the plane tangent to the cylinder at point B will be parallel to this plane [HDC] because the line of longitude extended from point B will be parallel to the axis, and the tangent at [point] B [along the line of longitude dropped through it] will be parallel to CD. Therefore, the plane containing lines LE and ET is not parallel to plane HDC. It will therefore intersect that [plane HDC]. Let it intersect along line LC, and draw line TC, which will of course be tangent [to the cylinder], since plane LET is tangent [to it, by construction]. Moreover, when line TD is drawn, angle CTD will be right because TD is a [a radial segment of the] diameter [of circle BTO, and CT is tangent to the circle at its endpoint].

[5.7] Now at point E form a circle, i.e., ESP, on the cylinder parallel to the base. Let K be the point on the [cylinder's] axis [where it intersects] this circle, and draw line KE. Draw line DL, as well, and it will certainly

intersect the plane of circle ESP. Let it intersect at point F, wherever that point may lie either outside or inside the circle, and draw lines KF and EF.[67] Then from point F draw FM perpendicular to the plane of circle BTO, and draw line TM.

[5.8] It is evident that KD is parallel and equal to FM [since they are perpendicular to parallel planes], and so KF is parallel and equal to DM. Likewise, KD is parallel and equal to ET, and KE is parallel and equal to DT. Hence, TE will be parallel and equal to FM, and so EF [will be] parallel and equal to TM.

[5.9] But plane KDL is perpendicular to plane BEO of the [cylindric] section, and it is perpendicular to the plane of circle ESP. Therefore, it is perpendicular to common section EF of the [cylindric] section and the circle. So angle EFK is right. Likewise, angle TMD is right [since DM and KF are parallel, as are TM and EF].

[5.10] Therefore, since angle DTC is right [by construction], rectangle DM,MC = rectangle TM,FE,[68] but since FM is parallel to CL [because CL is necessarily parallel to TE, given that it is the common section of planes TCLE and HDT, which are both perpendicular to the plane of circle ESP], then [triangles DFM and DLC will be similar and will have corresponding sides proportional (by Euclid, VI.4), from which it follows that] DF:FL = DM:MC. But DF > DM, so FL > MC. Consequently, rectangle DF,FL > rectangle DM,MC, so, since TM = EF, rectangle DF,FL > rectangle EF,FE [which = rectangle TM,FE, which = rectangle DM,MC], so angle LED > a right angle, for if it were a right angle, then because line EF is perpendicular to LD, rectangle DF,FL would be equal to EF^2.[69] It therefore follows that angle DEQ [adjacent to angle LED] is acute. Hence, the perpendicular [EU] dropped from point E, that perpendicular being perpendicular to tangent QL, will fall outside line ED and will intersect normal BD outside point D, which is what was set out [to be proven].[70]

[5.11] Now that these things have been set out, it is time to get to the proposition.

[5.12] **[PROPOSITION 17]** Assume a cylinder [in figure 6.5.17, p. 126], and let TH be a [visible] line parallel to the [cylinder's] axis [ZK]. TH will of course be parallel to the line of longitude [AG in the same plane with TH and the axis] of the cylinder.

[CASE 1]

[5.13] Therefore, if the center of sight [E] lies in the same plane as the axis and line TH, the [form of the] line can be reflected, and the reflection will occur from the line of longitude on the cylinder, which is the common section of the plane containing the center of sight and the axis and the

surface of the cylinder, as was shown in [proposition 28 of] book 5. Line TH will thus appear as a straight line [T′H′] because any normal dropped from a point on line TH [such as TT′ or HH′] will lie in the same plane as the center of sight and the axis, and it will be proven that the image of line TH is straight, just as it is proven for [straight] lines seen in plane mirrors.

[CASE 2]

[5.14] Let the center of sight [E in figure 6.5.17a, p, 126] lie outside the plane containing line TH and the axis, and [let] TH be parallel to the axis, which is ZK. Project a plane that passes through the center of sight and cuts the cylinder's surface parallel to the base. It will of course cut a circle [on that surface]. Let that circle be BF. [The form of] some point on line HT is reflected to the center of sight from some point on this circle. Let it be [reflected] from point B, and let E be the center of sight.

[5.15] Let Q be the point on line TH [whose form is reflected to E from B], draw lines EB and QB, draw line of longitude ABG from point B, and draw the normal ML through point B that intersects the axis at point L. Then from point E extend line EO parallel to ML, and extend QB until it intersects [EO]. Let the intersection be at point O.

[5.16] It is obvious that angle QBM = angle EBM [by construction], but angle QBM = [alternate] angle BOE because LM is parallel to OE [by construction]. Likewise, angle MBE = angle BEO, since [it is] alternate [given the parallelism of ML and EO]. Therefore, angle BOE = angle BEO, so BO and BE are equal [in isosceles triangle BEO].

[5.17] Now choose another point on line TH, let it be point T, and draw line TO. It is clear that line TH is parallel to line of longitude AG [by construction]. Therefore, they lie in the same plane, and line QBO lies in that plane, so line TO will lie in [that] same [plane]. Hence it will intersect line AG. Let it intersect at point G, and draw line EG.

[5.18] It is also clear that line AG is perpendicular to the plane of circle BF, as is the axis [ZK] to which it is parallel, and its plane [is] EOBF [which] cuts the cylinder parallel to its base. Thus, angle GBO [is] right, and [so] angle GBE is right. Consequently $GO^2 = GB^2 + BO^2$ [by Euclid, I.47]. Likewise, $GE^2 = GB^2 + BE^2$, and since BE and BO are equal [by previous conclusions], while GB is common, GO = GE. Hence, angle GOE = angle GEO [in isosceles triangle GEO].

[5.19] Moreover, if normal ZGN is drawn, it will be parallel to EO, since it is parallel to MBL [to which EO was made parallel by construction]. Therefore, angle TGN = [alternate] angle GOE, and angle NGE = [alternate] angle GEO, so angle TGN = angle NGE. Furthermore, since E, O, N, G, and Z lie in the same plane, and since G lies in that plane, E, G, and T will lie in the same plane, and so lines EG, NG, and TG lie in the same plane [which

is therefore the plane of reflection]. Thus, [the form of] T is reflected to E from point G.

[5.20] Furthermore, if point H is taken on line TH at the same distance from point Q as point T, and if line HO is drawn, it will pass through [some] point on line AG. Let it pass through point A. When normal DA[Z′] and lines EA and HAO are drawn, it will be a matter of proving as before that the two angles ABO and ABE are right, that the two sides AO and AE are equal, and that the two angles HAZ′ and EAZ′ are equal. And so [the form of point] H is reflected to [point] E from point A. Likewise, if any [other] point on line TH is chosen, it will be a matter of proving that [its form] is reflected to E from another point on line AG, so [the whole form of] line TH is reflected from line of longitude AG.

[5.21] **[PROPOSITION 18]** It remains to demonstrate that the image of line TH is curved. It is clear from the preceding [theorem] that [the form of] Q [in figure 6.5.18, p. 127] is reflected to E from point B, which is a point on circle [FB]. But since it is reflected in this way from the circle, if a line is drawn from point Q to the center of that circle, it will meet the normal [MBL] dropped from point B, and the intersection [of these two lines] will lie at a point on the axis. So draw QL intersecting ML at point L on the axis, and [this] is the center of circle FB. Then extend EB until it meets QL. Let the intersection be at point C. C will be the image of Q, and C lies in the same plane with lines QH and the axis [ZK], and [with] line of longitude AG.[71]

[5.22] It is also evident that [the form of] T is reflected to E from a point on a [cylindric] section of the cylinder, namely, point G. Moreover, [as established in proposition 16, lemma 5] a line can be drawn from point T perpendicular to a line tangent to another point on the [cylindric] section, and it will intersect normal NGZ dropped from point G outside the axis, that is, outside point Z, which is the intersection of normal NZ and the axis, for if line TZ is drawn, angle TZN will be acute [as stipulated in proposition 16]. Accordingly, draw TX [normal to the cylindric section, as prescribed, and] intersecting NZ at point X, and extend EG until it intersects TX at point I. I will be the image of point T.

[5.23] Likewise, when the line [HP] orthogonal at a point on the [cylindric] section from which reflection [of the form of point H] occurs is drawn from H, it will intersect normal DAZ′ outside of point D, which is a point on the axis. Let it intersect at point P, and extend EA until it intersects HP at point S. Point S will be the image of point H. Now draw line SI.

[5.24] It is clear that, since line TI intersects normal NZ, which is parallel to line EO, it will intersect line EO. The same holds for line HS; because it intersects normal DAZ, which is parallel to EO, it will intersect EO. But

since T's location with respect to point E is equivalent to and the same distance [from E] as H's location [by construction], the location of point T and of point H [will] likewise [be] equivalent with respect to point O, and [that of] points I and S is also equivalent with respect to O. The location of lines TI and HS will also be equivalent with respect to line EO.[72]

[5.25] Lines TI and HS will therefore intersect at the same point on line EO [since each lies in a plane with it, and both are inclined toward one another]. Let them intersect at point U. TUH will [therefore] be a triangle, and [straight] line IS will lie in the plane of this triangle. The axis, however, does not lie in this plane [since normals TIU and HSU bypass it].

[5.26] But TH does lie in the same plane as the axis so that plane [TZKH] intersects the plane of the triangle [TUH] along common section TH, not along some other [line of section]. Therefore, since point C lies in the plane of line TH and the axis, but not on line TH [itself], it does not lie in the plane of triangle TUH, whereas the two points I and S do lie in the plane of that triangle, so line ICS is curved, and the image of line TH will [therefore] be curved, which is what was set out [to be proven].

[5.27] But its curvature is slight because the perpendicular dropped from point C to the plane of the circle is extremely small,[73] and the closer the visible line is to being parallel to the line of longitude on the mirror, the less sharply curved it[s image] is, [whereas] the farther [it is from such parallelism] the more [sharply curved its image is].[74]

[5.28] **[PROPOSITION 19]** Furthermore, if line TH intersects the plane containing the center of sight and the axis, and if it is orthogonal to it, the center of sight will either lie in the plane of line TH intersecting the plane of the axis and the center of sight orthogonally, or [it will lie] outside [that plane].

[CASE 1]

[5.29] If [the center of sight] lies in that plane, it will lie beyond or in front of line TH. If [it lies] beyond, then, since that line has bodily dimensions, it will block the mirror from the center of sight, and so it[s form] will not be reflected [to the eye], although perhaps its terminal segments will appear and be reflected from the circle on the cylinder that forms the common section of the plane of line TH that cuts the cylinder and the cylinder [itself]. And the image of these terminal segments will [appear] just as [they do] in convex spherical [mirrors, as described in proposition 14, paragraph 4.138].[75]

[5.30] Likewise, if the center of sight lies in front of TH, part of that line will be hidden by the head containing the center of sight. Nonetheless, the visible part of the line is reflected [to the center of sight] from the circle

[formed by the plane of reflection] in exactly the same way as in convex spherical [mirrors, according to proposition 14, paragraph 4.138].

[CASE 2]

[5.31] But if the center of sight lies outside the plane of TH that cuts the plane of the center of sight and the axis orthogonally, then let E [in figure 6.5.19, p. 128] be the center of sight [above line TH] and XZG the cylinder.[76] [The form of point] H is reflected to E from some point on the cylinder. Let [it be reflected] from B. Let T lie the same distance [as H] from point E. I say that [the form of point] T is reflected to E from another point [i.e., other than B] on the cylinder, and that, since points H and T are equivalently situated and the same distance from point E, their points of reflection, i.e., B and G, will be equivalently situated and the same distance from point E. Therefore, the two points B and G will lie on a circle.

[5.32] Let BZG be the circle, with D its centerpoint. Draw lines HB, BE, TG, and GE, and from the centerpoint [D] draw normals to the tangents at B and G, i.e., [normals] DBO and DGS. Then draw line ED, and extend HB and TG until they intersect line ED.

[5.33] Since points H and T are equivalently situated and the same distance with respect to E and with respect to D, and since, by the same token, points B and G are equivalently situated with respect to D and with respect to E, lines HB and TG will be equivalently situated with respect to line ED, and so they will intersect at the same point on that line. Let [that intersection] be at point L.

[5.34] Produce the line of longitude on the cylinder that contains point Z, let this line lie in the plane containing the center of sight and the axis, let it be AZ, and draw [lines] LZN and DZC. Let Q be a point on line TH, that is, the point [on it] that lies in the plane of the center of sight and the axis, and from point Q draw a line parallel to line DZC. This line will fall on the axis, and LZN will fall on this line beyond point Q. Let it fall at point N.

[5.35] It is clear from the foregoing that angle [of incidence] HBO = angle [of reflection] OBE [by construction]. But angle HBO = angle LBD because they are vertical [angles], and angle OBE = the two angles BED + BDE because it is external [to triangle BDE and therefore equal to the two opposite interior angles, by Euclid, I.32]. Therefore, angle LBD = the two angles BED + BDE. So form angle MBD = angle BDE. It follows that angle MBL = angle BEL [i.e., BED], so the rectangle EM,ML = BM^2.[77]

[5.36] Draw line MZ. Since angle BDM > angle ZDM, and since the two sides ZD and DM [of triangle ZDM] are equal, respectively, to the two sides BD and DM [of triangle BDM], MB > MZ, so the rectangle EM,ML > MZ^2.[78] Let the rectangle EM,MI = MZ^2 [which means that EM:MZ = MZ:MI] and draw lines IB and IZ. Hence, angle MZI = angle ZEI [because triangles

MZE and MZI are similar according to the proportionality of sides EM and MZ in triangle MZE and sides MZ and MI in triangle MZI], so [angle] MZL > angle ZED.

[5.37] But since angle MBD has been posited equal to angle BDM [by construction], line MD = line MB [in isosceles triangle DMB]. But MB > MZ [by previous conclusions], so MD > MZ. Therefore, angle MZD > angle MDZ [since it is subtended by a longer line], so angle DZL > the two angles ZDE + ZED.[79] But angle DZL = [vertical] angle NZC, and [exterior] angle CZE = the two [interior and opposite] angles ZDE + ZED [in triangle ZED], so angle NZC > angle CZE.

[5.38] Let [angle FCZ] equal [to angle CZE] be cut [from NCZ] by line FZ, which will intersect line NQ [at point F] beyond point N. Therefore, since angle FZC = angle CZE, [the form of] F is reflected to E from point Z. [The form of point] Q is reflected to E from a point on the line of longitude passing through Z, that is, from a point on AZ beyond [i.e., below] Z. For if [it occurs] from a point this side of Z, i.e., nearer E, the line extended from point Q to that point of reflection will cut line FZ, and so the point of intersection is reflected to E from two points, which is impossible.[80]

[5.39] Take point K below Z from which [the form of] Q is reflected to E, then, and extend line EK until it intersects line NQ [i.e., the cathetus dropped from object-point Q] at point P. P will be the image of Q. But [the form of] H is reflected to E from a point on the cylindric section [formed on the cylinder by plane of reflection HBE]. Therefore, if a normal is dropped from point H to the line tangent to the [cylindric] section at some point [on it], that normal [i.e., the cathetus] will intersect normal CZD outside the axis [by proposition 16, lemma 5]. Let it intersect at point U.

[5.40] Likewise, from point T a normal can be drawn to the [cylindric] section from a point on which [its form] is reflected to E. And since points H and T are equivalently situated with respect to line CZD, the same also holds for the points on the [cylindric] section through which the normals [i.e., the catheti] pass, so those two normals will intersect at the same point on line CZD. Accordingly, let them intersect at point U.

[5.41] [Therefore, the extension of] line EB will intersect line HU. Let R be the point of intersection. By the same token, let EG intersect TU at point Y, and draw line RY.[81] It is obvious that R is the image of H, [and] Y is the image of T, and we have triangle ERY. Point Z lies outside the plane of this triangle, so the plane of this triangle is higher than line EP, and so P lies outside [that plane]. Hence, line RPY will be curved, and it is the image of line TH, and the curvature of this image is certainly not inconsiderable, which is what was set out [to be proven].

[5.42] It is therefore clear that in these [sorts of] mirrors, if a straight visible line is parallel to a line of longitude on the cylinder, its image will be either straight or verging toward straightness. But if a straight visible line is parallel to the width of the mirror [i.e., the plane through it is perpendicular to the cylinder's axis], its image will be curved, and its curvature will not be inconsiderable.

[5.43] Furthermore, among [visible] lines oriented between these two [extremes], the images of those that verge more closely toward an orientation parallel to the longitude of the cylinder will be closer to straight, whereas the images of those that are nearer to an orientation parallel to the [cylinder's] width will be more curved. And the curvature of the images will diminish or augment depending on how close or far the lines are from either of these orientations, and this is what was set out [to be demonstrated].

[CHAPTER 6
On Convex Conical Mirrors]

[6.1] Moreover, in convex conical mirrors the same errors occur as happen in convex cylindrical [mirrors],[82] for [straight] visible lines that are parallel to the longitude of the cone appear straight, whereas those parallel to the width [of the cone appear] curved, and for those at intermediate positions, their curvature augments or diminishes according to how near or how far [they are from those extreme positions], and this will of course be demonstrated. However, we must set forth something beforehand, and it is [as follows].

[6.2] **[PROPOSITION 20, LEMMA 6]** If a point of reflection is taken on the surface of a cone and a [conic] section is produced [on that surface] to pass through that point, and if a point is taken on that [conic] section farther from the vertex of the cone than the point of reflection and a normal is dropped from the selected point to a line tangent to the [conic] section, this normal will intersect the normal dropped from the point of reflection [at a point] outside the axis.

[6.3] For instance, let ABGZ [in figure 6.6.20, p. 129] be a cone standing upright on its bases [i.e., a right cone], A the cone's vertex, BFZ the [conic] section [produced on its surface], E the point of reflection, and Z the point on the [conic] section farther from [vertex-]point A than E. At point Z let there be a plane cutting the cone parallel to its base. It will of course cut it along a circle [forming the] common [section of the cutting plane and the cone's surface]. Let that circle be GBRZ, draw lines AZ and AE, and extend

AE until it is equal to AZ. It will reach the circle. So let it fall at point O on it.

[6.4] Let C be the center of the circle, draw axis AC, and from point E draw the normal [ED] to the plane tangent to the cone [at that point]. It will of course intersect the axis in the vicinity of the circle's centerpoint C. Let [that intersection] be at point D, and draw line DZ.[83]

[6.5] Then from point O draw a normal [OK] intersecting the axis at point K, and draw lines DZ and KZ. At point Z produce [line] TQ tangent to the [conic] section, and [at the same point produce] another [line] ZY tangent to circle BGZ.

[6.6] Next draw line BCZ, and from point C draw CR perpendicular to line BCZ. It will of course be perpendicular to the axis, since the axis is perpendicular to the circle [in whose] plane [CR lies], so CR is perpendicular to plane ACZ. It will also be parallel to tangent ZY, so ZY is perpendicular to plane ACZ, [and] so TQ is not perpendicular to that same plane.

[6.7] However, because K is the [the endpoint of] pole [KC] in circle BRZ, then, since lines KO and KZ are equal [because they are lines of longitude on a right cone with its vertex at K and its base circle passing through Z and O], and since axis AK is common [to triangles AOK and AZK], angle AOK = angle AZK, and so angle AZK is right [since angle AOK was constructed as a right angle]. Therefore, since line KZ is perpendicular to [line] AZ, which is a line of longitude [on the mirror], it will be perpendicular to the plane tangent to the cone['s surface] along this line of longitude. But TQ lies in the [same] tangent plane because it is the common [section] of the tangent plane and the [conic] section. Accordingly, KZ is perpendicular to TQ.

[6.8] Furthermore, draw HZ in the plane of the [conic] section perpendicular to line TQ [and therefore normal to the section itself]. Since line KZ lies outside the plane of the [conic] section, it will intersect line HZ and will [therefore] not form a single line with it. Hence, plane KZH intersects the plane of the [conic] section along common section HZ, and it intersects line TQ at point Z. In addition, plane AZK intersects plane AZH along common section KZ.

[6.9] But DZ lies in the plane of the [conic] section, and it is intersected by line KZ at point Z, and point T [lies] above plane KZH, point Q below [it, i.e., on the other side of it from T]. And so plane KZH cuts plane DZQ along a common section, and that common section is perpendicular to line TQ because that line lies in plane HZK to which TQ is perpendicular. And since plane HZK intersects plane DZQ, and since plane HZK slants in the direction of [segment] ZE [of the conic section], the common section [HZX] of those planes will lie between lines QZ and DZ, and so it will intersect normal ED [at point X] outside the axis. That it [i.e., the normal to the

section at Z] must necessarily intersect it [i.e., normal ED dropped from center of sight E] has been demonstrated in book 5, proposition 26,[84] and so what was set out [to be demonstrated has been shown].[85]

[6.10] **[PROPOSITION 21]** So let there be a [right] cone [in figure 6.6.21, p. 129] with its vertex at A, AH being its axis and AZ a line of longitude, and from point Z to the plane tangent to the cone along line AZ drop a perpendicular, which will perforce intersect the axis [at point H]. Let it be line TZH.

[6.11] From point A extend line AN outside the cone [and] above the plane tangent to the cone along line AZ so that it forms an acute angle with the axis, as well as with line AZ of longitude. From point H within plane AHN, draw line HO forming an angle [AHO] equal to angle AHZ, that line necessarily intersecting line AN [at point O. Consequently] when a circle is produced through point Z parallel to the [cone's] base, HO will pass through [that] circle just as HZ passes through it.

[6.12] Now draw line OZ, and extend it to point F. Since line OZ lies above the plane tangent to the cone along line AZ, then because HZ is perpendicular to that plane [by construction], angle OZH will be greater than a right angle [because it intersects the plane tangent to AZ from above]. Consequently, [adjacent] angle FZH is acute [so that ZF lies inside the cone].

[6.13] From point Z draw ZM tangent to the circle, and from point F draw a line perpendicular to AZ to fall at point E on it, and when it is extended, it will intersect AO because angle OAZ is acute [by construction, and because AO, AZ, and OZF lie in the same plane]. Accordingly, let it fall at point N, and from point E draw line QE parallel to line TH.

[6.14] Then from point E draw LE parallel to line MZ. It is evident that MZ is perpendicular to AE because it is perpendicular to TH, as well as to the diameter of the circle, to which it is tangent. Therefore, LE is perpendicular to AE [since it is parallel to MZ, by construction].

[6.15] Now produce plane LQD cutting the cone. It will of course form a conic section [because the cutting plane intersects the axis below the circle passing through E]. Hence, since AE is perpendicular to FN [by construction], as well as to QD and LE, FN will lie in that plane [QEDL], which cuts the cone [along the aforementioned conic section]. Accordingly, produce CF parallel to QE. It will of course be parallel to TZ.

[6.16] But since angle OZT is acute [by previous conclusions, adjacent] angle TZF is obtuse. From point Z draw a line [ZC] that forms with TZ an angle [TZC] equal to angle OZT, and that line will necessarily intersect FC. Let it intersect at point C, and draw line EC. Therefore, since CZ and OZ

lie in the same plane, and since angle OZT = angle TZC [by construction, the form of] point O is reflected to C from point Z.

[6.17] Since, however, angle OZT = [alternate] angle ZFC, and since angle OZT = [alternate] angle ZCF, sides ZC and ZF [of triangle ZCF] will be equal, and given that angle FEZ is right [by construction], $FZ^2 = EZ^2 + EF^2$ [by Euclid, I.47], while $CZ^2 = EZ^2 + EC^2$ [by the same theorem]. Therefore, CE and FE are equal, and so angles ECF and EFC are equal, whereby angles NEQ [which is alternate to EFC] and QEC [which is alternate to ECF] are equal. And since C, E, and N lie in the same plane [i.e., the plane producing the conic section through point E, the form of] point N is reflected to C from point E.

[6.18] Likewise, draw some line from point F to some point on line ZE, and extend it to ON. As to the point on line ON where it falls, it will be proven that [its form] is reflected to C from the [corresponding] point on ZE because it cuts that line. In the same way, as well, for all such lines, the proof will take its start from perpendicular FE with respect to line EZ, which will be the common terminal [base-line], and so [the form of] any given point on line ON is reflected to C from some point on line EZ.

[6.19] **[PROPOSITION 22]** Having demonstrated this point, we should state [that], when the eye perceives straight lines that pass through the vertex of a right convex conical mirror at a slant to the mirror's axis, the forms of those [lines] will be somewhat convex in that mirror.

[6.20] Accordingly, let the right conical mirror be ABG [in figure 6.6.22, p. 130], with A as its vertex and AD as its axis, let us produce line AZ at random on its surface, and let point Z be marked on it at random. Let a plane pass through Z parallel to the base of the cone, and let it form circle ZU. Then from Z let us drop ZH perpendicular to AZ. Hence, this line will intersect the cone's axis, so let it intersect [that axis] at H.

[6.21] From Z let us extend line ZM tangent to the circle [ZU], and let us extend a line from A that forms an acute angle with both lines AZ and AH, and let it lie outside the plane tangent to the cone that passes along line AZ, which is possible. Let it be AO, then, and let us produce a line from point H within the plane containing AO and AH that forms an angle [AHO] with AH equal to angle ZHA. This line [HO] will therefore intersect AO because the two angles at A and H [i.e., OAH and AHO] are acute. So let them [i.e., AO and HO] intersect at O.

[6.22] Accordingly, line HO will intersect the circumference of circle ZU because angle AHO = angle AHZ [by construction]. So let it intersect at U, and let us extend AU in a straight line. Let us also extend perpendicular HZ to T, let us produce OZ and extend it directly to F, and extend AZ

to E. Therefore, angle FZH will be acute because line OZ cuts the plane tangent to the cone and passing along AZ. Hence, line FZ lies below the common section of plane OZH and the [aforementioned] tangent plane [passing along AZ], and this common section forms a right angle with line HZ. Thus, angle OZH is obtuse, [and] so [adjacent] angle FZH is acute.

[6.23] So take point F on ZF, from it extend FE perpendicular to AE, and continue it in a straight line. It will therefore intersect line AO because angle OAE is acute. Let it intersect at N, then, and extend line ED from E parallel to line ZH. ED will thus be perpendicular to the plane tangent to the cone passing along AE.

[6.24] Then from E draw line EL parallel to line ZM, and produce the plane containing LE and ED. It will therefore intersect the surface of the cone and will form a [conic] section, for this plane is oblique to axis AD.

[6.25] Let the [conic] section be BEG′. MZ is perpendicular to plane AZH [since it was constructed tangent to the circle at point Z and is therefore perpendicular to the diameter passing from Z through axis AH of the cone], and this was established earlier [in proposition 21]. Therefore, line LE is perpendicular to plane AED, so angle AEL is right, angle AEN is right, and likewise angle AED is right [all three by construction]. Consequently, lines LE, NE, and DE lie in the same plane [to which AE is perpendicular]. Hence, line FEN lies in the plane of the [conic] section [BEG′].

[6.26] From point F extend line FR parallel to line DE. Accordingly, this line will be parallel to line HZ [to which DE was constructed parallel]. In plane OZH draw a line [ZR] from Z that forms with ZT an angle equal to OZT. This line will therefore intersect FR because it will intersect ZH, which is parallel to FR [by construction] and lies in the same plane with it, since ZF lies in that plane. So let it intersect at R.

[6.27] Accordingly, the two angles at R and F [i.e., ZRF and ZFR] are equal, for they are equal to the two angles [OZT and TZR] at Z.[86] So the two lines RZ and FZ [within isosceles triangle ZRF] are equal. But it has been shown that line FEN lies in the plane of the [conic] section [BEG′], and line FR is parallel to line ED [by construction]. It [i.e., FR] therefore lies in the plane of the [conic] section.

[6.28] Let us then draw RE. It will thus lie in the plane of the [conic] section. Extend DE to K, and it has been shown that EA is perpendicular to the plane of [that] section. Hence, each of the angles AER and AEF is right, and the two lines FZ and RZ are equal. Consequently, the two lines RE and FE are equal, so the two angles ERF and EFR are equal.

[6.29] Accordingly, the form of N will be reflected to R from E, and the form of O will be reflected to R from Z. Moreover, every line extended from F to some point on line AN will intersect AE. But it is clear that that line

will be equal to the line extended from R [to the same point on AE where the line from F intersects it] because AE is perpendicular to the plane in which lines RE and FE lie, since this plane is the plane of the [conic] section, and the two lines RE and FE are equal. Hence, both lines extended from R and F to a given point on line AE are equal.

[6.30] It is therefore evident that the form of [any] point on AN will be reflected to R from a point [such as] that on line AE. And the same holds for any point lying on AN beyond N; if it is connected with F by a straight line, that line will intersect AE beyond E. It is also evident that the form of a point on AN will be reflected to R from a point on AE. From this, therefore, it is evident that the form of line AN, as well as any [line] continuous with it, will be reflected to R from a straight line on the surface of cone ABG, and the same holds for every line extended from A at a slant to the cone's axis.[87]

[6.31] Let us draw ND [in figure 6.6.22a, p. 131, abstracted from figure 6.6.22].[88] It will therefore intersect the periphery of the [conic] section because the two points N and D lie in the plane of [that] section, and N lies outside the section['s periphery], whereas D lies inside the section['s periphery]. So let it intersect the periphery of the [conic] section at C, and since triangle AOH lies in the same plane, ND will lie in the same plane as triangle AOH.

[6.32] [Point] C is therefore in the plane of triangle AOH, and the two points A and N lie in the plane of this triangle. Hence, points A, N, and C lie in the plane of triangle AOH. But points A, U, and C lie on the surface of the cone. Accordingly, points A, U, and C lie on the common section of the surface of the cone and plane AND. But this common section is a straight line. So points A, U, and C lie in a straight line.

[6.33] Extend AU directly to C, then, and extend RZ in a straight line. Accordingly, it will intersect OH [because O, Z, R and H all lie in the same plane of reflection]. Let it intersect at P. P therefore lies in the plane of triangle AOH. So extend AP, and let it continue in a straight line. It will therefore intersect ND at G, and since F lies below the plane tangent to the cone that passes along line [of longitude] AZE, angle FED will be acute, whereas [adjacent] angle DEN is obtuse. Hence, [interior] angle ENC [of triangle NED] is acute [because it is smaller than opposite exterior angle FED, which is acute].

[6.34] Furthermore, let line CZ′ be tangent to the [conic] section. It is clear, then, as [shown] in an earlier proposition, that angle DCZ′ is obtuse[89] and that the perpendicular erected on CZ′ at C cuts angle DCZ′ and will intersect ED beyond D. So this perpendicular will intersect ED at S.

[6.35] Hence, the perpendicular extended from N to the line tangent to the [conic] section [at the point where that perpendicular intersects the

conic section] will intersect the [conic] section beyond C, that is, farther from E than C [lies from it], for these perpendiculars [i.e., NQ and ED] will intersect outside the periphery of the [conic] section [i.e., on the other side of that periphery from N]. Hence, the perpendicular extended from N to the line tangent to the [conic] section will not cut angle DCZ'. It will therefore lie farther from NE than CD [does], and this perpendicular cuts ED beyond D.

[6.36] So let the perpendicular dropped from N to the line tangent to the [conic] section be NQ. Also, RE intersects EN, and it intersects the periphery of the [conic] section and lies in its plane, while NQ [also] lies in the plane of the [conic] section. Hence, if RE is extended in a straight line, it will intersect NQ. Let it intersect at Y, then.

[6.37] Plane AND intersects the plane of the [conic] section. Since point E lies outside plane AND because plane AND is not [in] the plane of the [conic] section [whereas E is], and because A lies outside the plane of the [conic] section, since AE is perpendicular to the plane of [that] section, whereas E lies on its periphery, then ND is the common section of plane AND and the plane of the [conic] section, and NQ will intersect [that] section beyond C [i.e., on the opposite side from C and E]. Hence, NQ lies outside plane AND. Y therefore lies outside line APG.

[6.38] Accordingly, if the center of sight lies at R, and if line AON lies on some visible object, then P will be the image of O, Y will be the image of N, and A will appear at its [actual] location, since it lies at the vertex of the cone.[90] And the image of line AON will be the line passing through points A, P, and Y, but this line is convex because it lies outside [straight line] APG.

[6.39] So let that [image-]line be APY, and it has already been shown that the forms of all the points on AN are reflected to R from AE. Therefore, the radial lines according to which those forms are reflected lie in the plane of triangle RZE, so all the images of [the points on] line AN lie in that plane.

[6.40] Hence, convex line APY lies within that plane, and P lies closer to R than Y does, and the convexity of this image will be toward the center of sight, and it will [therefore] be of slight [apparent] convexity.[91] Moreover, the [length along the] cross-section [AY] of this image will be slightly smaller than the line itself [i.e., AN of which it is the image]. Consequently, the images of straight lines that are extended from the vertex of the cone at a slant to the axis are perceived by sight as convex in such a mirror, and the forms of these lines are reflected from straight lines among the lines extended along the cone's longitude, and this is what we wanted to prove.

[6.41] On the other hand, the forms of lines that are parallel to the width of a conical convex mirror are reflected from convex lines on the mirror's surface, and the convexity of these lines is obvious, as [it is] in a convex cylindrical mirror, and for the same reason, and it will likewise be evident that the images of these lines will be quite convex and manifestly [so] to the [visual] sense. Also, the center of sight will lie outside the plane that contains the convexity of the forms of these lines, and the cross-sections of the images of these lines will be considerably shorter than the lines themselves.

[6.42] As to lines that are slanted between these two extremes, however, those whose orientation approaches that of lines extended along the length of the cone have slightly convex forms, whereas those that approach lines parallel to the width of the cone have forms that are clearly convex.

[6.43] But also, curved lines that approach the vertex of the cone have smaller, narrower, and more convex forms, whereas those that approach the base of the cone have larger forms, according to what was demonstrated for convex spherical mirrors—i.e., that the smaller the mirror, the smaller the circles that fall on its surface—and so the images [falling on those smaller circles] will lie closer to the center [of curvature], from which it follows that they will be smaller.

[6.44] By the same token, sections that lie on a conical mirror toward the cone's vertex are narrower and shorter [than those that lie farther from it], and so the image [within such a section] will be nearer the point where the normals dropped from the visible line to the lines tangent to the sections, which form the common section [of the plane of reflection and the plane tangent to the mirror at that point], intersect, and so those images will be smaller.

[6.45] On the other hand, the opposite holds for sections that lie toward the base of the mirror, so it happens that a form perceived in a convex conical mirror will take on a conical form, i.e., what lies toward the vertex of the mirror will be narrower, whereas what lies toward the base will be broader, and the convexity of a form along the width [of the mirror] will be evident.

[6.46] It also happens in these [sorts of] mirrors that the closer the visible object approaches the mirror, the larger it will appear, whereas the farther away it will be, the smaller it will appear.

[6.47] Therefore, the misperceptions that occur in these sorts of mirrors are in every way like those that occur in convex cylindrical mirrors except for the conical shape of the form. And without exception the form of a visible object that is perceived by reflection will always take the shape of the surface of the mirror from which the form is reflected, and the reason

for this is that the image-location is invariably determined by the shape of the mirror's surface and by the place where the normals intersect, so the [shape of the] mirror's surface always plays a role in the shape [of the image] of the visible object that is perceived in the mirror. However, the compound misperceptions [arising] in this [sort of] mirror are identical to the [compound] misperceptions [occurring] in the previously discussed mirrors [i.e., convex spherical and convex cylindrical].[92]

CHAPTER 7
Concerning the Misperceptions That Occur in Concave Spherical Mirrors

[7.1] In these [mirrors], in fact, more [misperceptions] occur than in all the convex and plane mirrors,[93] for what occurs in the latter occurs in these as well—i.e., a weakening of light and color and a variation in orientation and distance—for it is reflection alone, not the shape of the mirror, that causes this [sort of variation]. [But] in addition, there is more variation in [image] size in these mirrors than in convex mirrors, for in convex [mirrors] an object will generally be perceived as smaller [than it actually is], whereas in concave [mirrors] it will sometimes be perceived as larger, sometimes as smaller, [and] sometimes as it actually is, and this happens according to how it changes position with respect to the mirror as well as to the center of sight, as we will demonstrate in this chapter.

[7.2] It also happens in these mirrors that a single visible object may appear as two, or three, or four, and this is not the case in plane and convex mirrors, for in those [kinds] a single visible object is perceived only singly, whereas in concave [mirrors, such is] not [the case].

[7.3] Furthermore, the arrangement of the visible object's parts is perceived in convex and plane mirrors as it actually is, whereas in spherical concave [mirrors it is perceived] otherwise in several situations,[94] and this in two ways: namely, in convex spherical mirrors there is no deception in the perception that a single thing is single and the perception of the arrangement of its parts according to how it actually is, and since there is deception in regard to these aspects in spherical concave mirrors, it is clear that nothing is perceived in these mirrors without deception, either invariably or at some time according to variation in the position [of the object vis-à-vis the mirror as well as the center of sight].

[7.4] However, weakening of light and color as well as change in position and distance occur in these mirrors just as [they] invariably [occur] in the others, and they do so in every situation. But size, shape,

and number are subject to deception in these mirrors in some situations, as we will demonstrate.

[7.5] Concerning number it has been shown in chapter [2, book 5] on image [formation] that in concave spherical mirrors one object has one, two, three, or four images, and that the form of a visible object is always perceived at its [appropriate] image-location. However, one object perceived in concave spherical mirrors may be perceived as one, perhaps as two, perhaps as three, and perhaps as four, which does not happen in convex and plane mirrors.

[7.6] As to the arrangement of the visible object's parts, it has also been claimed in chapter [2, book 5] on image [formation] that the form of a single [object-]point is reflected from the circumference of a [great] circle [on the mirror's surface] and that visible objects whose images lie beyond or behind the center of sight, in front of it, or at the center of sight [itself] appear blurred and not clear, and anything of this sort does not have the arrangement of parts that the visible object itself has. And here, as well, what obtains in these mirrors is other than what obtains in convex and plane mirrors. But the reasons for this phenomenon have been discussed in the chapter on image [formation].[95]

[7.7] It thus remains [for us] to show that what is perceived in these mirrors may be perceived larger, smaller, or the same size [as the object itself], and that in certain situations it may be perceived inverted and in others erect, and that a straight object is perceived as concave, convex, or straight in mirrors of this sort, and that convex and concave objects are also perceived other than they [actually] are [in this sort of mirror]. And these [misperceptions] also arise from a variation in the arrangement of the visible object's parts, and we will demonstrate this in the following way.

[7.8] **[PROPOSITION 23]** Accordingly, let there be a concave spherical mirror centered on A [in figure 6.7.23, p. 132], let it be bisected by a plane passing through its center, let it form [great] circle BG, let a line [AU] be extended within it at random, and let it be bisected at O.

[7.9] Take A as a centerpoint, and at the distance of AO [as radius] let us form a circle, and let it be EZ. Choose some point T at random on line OU, and from T extend lines TN and TM perpendicular to line AU. Then from T extend lines TE and TZ tangent to circle EZ, and let us extend AE and AZ, and let them continue to B and G. Let us extend TB and TG, and let us draw BM parallel to AT and also GN parallel to AT, and let us connect AM and AN and extend them in straight lines.[96]

[7.10] Therefore since AO = OU [by construction], AE = EB, and AZ = ZG, and because TE is tangent to circle EZ, TE will be perpendicular to BA, and likewise TZ [will be] perpendicular to AG. Hence, line BT = [line]

TA [by Euclid, I.4], [line] TG = [line] TA, angle TBA = angle TAB [within isosceles triangle TBA], and angle TGA = angle TAG [within isosceles triangle TGA]. And since BM is parallel to AT, angle MBA = [alternate] angle BAT. Therefore, angle MBA = angle ABT, and likewise angle TGA = angle AGN.

[7.11] So when the center of sight is at T, and when M and N lie on some visible object, the form of M will be extended along line MB and will be reflected along BT, and the form of N will be extended along NG and will be reflected along GT. The center of sight at T will therefore perceive points M and N [at locations] beyond points [of reflection] B and G, and [so it will perceive the entire image of] line MN beyond arc BG.[97]

[7.12] Also, since TE is perpendicular to AB, angle ABT will be acute. But angle MBA = angle ABT. Thus, TB > BM, so AT > BM, and they [i.e., lines AT and BM] are parallel. Consequently, [line of reflection] TB will intersect [cathetus] AM. Let them intersect at F, then. F is thus the image of M, and it will be demonstrated equivalently that [line of reflection] TG will intersect [cathetus] AN. Let it intersect at Q, then. Q will thus be the image of N.

[7.13] Let us then connect FQ, which is the cross-section of the image of MN, and since TE and TZ are equal, angles TA[E]B and TAZ[G] will be equal, lines TB and TG will be equal, lines BM and GN [will be] equal, and lines AM and AN [will be] equal. Moreover [given the similarity of triangles AFT and MFB], AF:FM = AT:BM, and AF:FM = AT:GN = AT:BM [because GN = BM], so AF:FM [= AT:GN] = AQ:QN, and AM = AN. Hence, AF = AQ, so FQ is parallel to MN. Thus, FQ > MN. But FQ is the cross-section of the image of MN. Accordingly, if the center of sight is at T and MN lies on some visible object, the eye will perceive its form as larger than [object-line MN] is.

[7.14] **[PROPOSITION 24]** Now [in figure 6.7.24, p. 132] let us duplicate circle BG, line AT[U], and lines AB, AG, and TB [as given in figure 6.7.23]. Let TK be perpendicular to the plane of circle BG at point T, and let us draw KA, KB, and KG. Thus, planes KBA and KGA intersect the sphere [of the mirror] at its center perpendicular to [the appropriate] planes tangent to its surface.[98] Within these [planes], then, the form [of any given visible object] is reflected, and the two common sections between these two planes and [the surface of] the sphere form great circles from whose circumference the forms are reflected.

[7.15] Let us then draw BM parallel to AK in plane BKA, and let it be shorter than AK. Let us draw AM, and let it be extended in a straight line, and extend KB until they intersect at F. Then draw NG in plane KGA, let it be parallel to AK, and assume it is equal to BM. Let us connect AN, let

it be extended in a straight line, and extend KG in a straight line until they intersect at Q. Then let us connect MN and FQ.

[7.16] Accordingly, since BT = TA [as concluded in the previous proposition], BK = KA [by Euclid, I.4], and GK = KA [by Euclid, I.4]. Hence, BK = GK, angle KBA = angle KGA, and angle KAB = angle KBA. Likewise, angle KGA = angle KAG, so angle ABM [which is alternate to angle KAB] = angle ABK [which = angle KAB], angle AGN [which is alternate to angle KAG] = angle AGK [which = angle KAG], and angle ABM = angle AGN [since both are equal to equal angles KBA and KGA]. In addition, line BM = line GN [by construction]. Thus, line AM = line AN; so [line] AF = line AQ. The two lines FQ and MN will therefore be parallel, so [line] FQ > line MN.

[7.17] Hence, when the center of sight lies at point K, and when line MN lies on some visible object, the form of M will be extended along line MB and will be reflected along line BK in the plane of the circle passing through points B, A, and K, whereas the form of point N will be extended along line NG and will be reflected along line GK within the plane of the circle passing through points G, A, and K.

[7.18] And [so] point F will be the image of point M, while point Q will be the image of point N, and line FQ will be the cross-section of the image of [the entire line] MN. But we have already demonstrated [in paragraph 7.16] that line FQ > line MN, so when the center of sight is at point K, and when line MN lies on some visible object, the eye will apprehend the form of line MN on line FQ. Therefore, it will perceive the form [of the visible object] as larger than the visible object [itself].

[7.19] Accordingly, if we rotate the entire figure around line AU, while keeping [AU] itself stationary [to form the axis of rotation], point K will produce a circle that is perpendicular to line AU, and so every point beyond that point on that circle will be situated with respect to a line equivalent to line MN as K is situated with respect to MN.

[7.20] Consequently, if the center of sight lies at any point on the circumference of this circle, and if a line equivalent to line MN lies on the surface of some visible object [that is similarly disposed], the eye will perceive the form of that line [as] larger [than the line itself]. Likewise, moreover, if we extend TK in a straight line and take some point on it other than K [as a center of sight], and if we extrapolate at every stage from that point, which is equivalent to point K, the case will be like the case for point K.

[7.21] On the basis of these two propositions [i.e., 23 and 24], therefore, it is evident that in concave spherical mirrors many objects are perceived [as] larger [than they actually are] in many situations.

[7.22] **[PROPOSITION 25]** To continue, let AB [in figure 6.7.25, p.133] be a concave spherical mirror centered on E, and let us produce a plane passing through E, and let it form [great] circle AB [on the sphere]. Let us extend line EZ randomly from E to G, and from G let us drop GD perpendicular to the plane of circle AB, and let us mark point D on it at random. Then let us connect DE and extend it to O, let us produce EB so that it forms an obtuse angle [DEB] with ED, and let us produce EA so that it forms an angle [AED] with ED equal to angle DEB. Let us then connect DA and DB. Accordingly, the planes of the two triangles DAE and DBE intersect one another along line DE, and the two acute angles DBE and DAE will be equal.

[7.23] Now from B in the plane of triangle DBE let us produce a line [BO] forming an angle [EBO] with EB equal to angle DBE. Hence, this line intersects line DE, since angle BEO is acute, and the angle [EBO] at B is acute. So let it intersect at O.

[7.24] From A let us also produce a line [AO] in the plane of triangle DAE that forms an angle [EAO] with AE equal to angle DAE. So let it intersect DE at O because the two angles AEO and BEO are equal [by construction], and because the angles at the two points A and B [i.e., EAO and EBO] are equal [by construction].

[7.25] Let us then produce ET so that it forms a right angle with EB, and let us extend TE in the direction of E and BO in the direction of O, and let them intersect at H, and [so] TE = EH [insofar as triangles TEB and HEB are equal, by Euclid, I.26]. Let us likewise produce EK so that it forms a right angle with EA, let us extend it in the direction of E, and let us extend AO, and let them intersect at L. Therefore, KE = EL.

[7.26] Let us then connect TK and LH. They will thus be equal [because they lie within triangles KTE and HLE that are equal, insofar as KE = EL, TE = EH, and angle KET = vertical angle HEB]. Hence, if the center of sight lies at D, and if LH lies on some visible object, then D will perceive LH in mirror AB, and T will be the image of H [whose form is reflected from point B], K the image of L [whose form is reflected from point A], and so TK will be the cross-section of the image of LH, and it is equal to it.

[7.27] Consequently, if we rotate the entire figure, leaving HL stationary [as the axis of rotation], D will produce a circle, and if the center of sight lies at any given point on its circumference, it can perceive some visible object equivalent to line LH, and the image will be equal [in size] to it. And likewise, if the center of sight lies at O and TK is the visible object, the image will be the same size as the visible object.

[7.28] But yet, if the visible object is LH, if the eye is at D, and if TK is the image, the image will be inverted; [for] if H lies on the right [of object

HL from D's point of view its image] T will lie on the left [of image TK from that same point of view], whereas if H lies to the left, T will lie to the right, and if H lies above the line, T will lie below the line, and the same for L.

[7.29] Moreover, if the visible object is TK, if the center of sight is at O, and if the image is LH, the form will be erect, for image LH will lie beyond the center of sight, and it will be perceived ahead of the visible object, as was shown in chapter [2] on image [formation] in the fifth book, and the eye will perceive H, which is the image of T, along line BO, and L, which is the image of K, along LO.[99]

[7.30] It is therefore clear that an object is sometimes perceived in concave [spherical] mirrors the same size as it [actually] is.

[7.31] **[PROPOSITION 26]** Now let us continue BH [in figure 6.7.26, p. 133] in a straight line and mark R on it[s extension], and let us connect RE. Angle REB will therefore be obtuse [since it is larger than HEB, which is a right angle, by construction].

[7.32] Let us then extend RE to N. Hence, RB > BN [because BT = HB in isosceles triangle HBT, and BN < HT, while RB > HB]. Moreover, RB:BN = RE:EN [by Euclid, VI.3, since EB bisects angle NBR], so line RE > [line] EN.

[7.33] Let us also extend AL in a straight line [to M], and let AM = BR. Let us connect ME, and let it continue to U. Thus, ME > EU. Then let us connect MR and NU. MR will therefore be longer than NU.

[7.34] Accordingly, if MR lies on some visible object, and if the center of sight is at D, NU will be the cross-section of the image of MR, and NU < MR. On the other hand, if the center of sight is at O, and if NU lies on some visible object, MR will be the image of NU, and it is longer than NU.

[7.35] But if MR is the visible object, and if NU is its image [as viewed from D], then the image will be inverted, whereas if NU is the visible object and MR is its image [as viewed from O], the image will be correctly oriented, for if it lies beyond the center of sight, that image will appear ahead [of the object], and every point on the image will appear along a specific line among the [corresponding] radial lines.[100]

[7.36] **[PROPOSITION 27]** To continue, let us mark point Q on line OH [in figure 6.7.27, p. 134]. Let us connect QE, and let it continue to C. Let OF = OQ, and let us connect EF and let it continue to I. The two lines CE and EI will thus be longer than the two lines EF and QE, and [so] line CI > line FQ [in similar triangles EIC and QEF].

[7.37] Hence, if the center of sight is at O, and if CI lies on some visible object, FQ will be the image of CI, and FQ < CI. Moreover, FQ will appear

along the two lines AO and OB. Therefore, the form [of CI] will lie in front of the center of sight and will be smaller than the visible object [itself], and it will be properly oriented.

[7.38] But if the center of sight is at D, and if FQ lies on some visible object, CI will be the image of FQ. It is longer than FQ, and [so its] form will be inverted in front of the center of sight.

[7.39] So it is evident that in concave [spherical] mirrors the form of a visible object is perceived as smaller [than the object itself], larger [than that object], or the same size [as the object].

[7.40] **[PROPOSITION 28]** Now let AB [in figure 6.7.28, p. 135] lie [on] a concave [spherical] mirror [with] G its center, let that mirror be bisected by a plane passing through its center, and let it form [segment] AB [of a great] circle. Let us extend line GD at random, let it pass to E on the side of G, let the center of sight be at E, and let T lie on the surface of the eye.

[7.41] Then let us draw TH perpendicular to line ED, let ZT = TH, and let [the center of sight at] E perceive [the form of] H [by reflection] from A.[101] Consequently, the two points A and H will lie on opposite sides of point G, for if they lay on the same side, the line extending from the mirror to A would not cut the angle that the two radial lines [of incidence and reflection] form.[102]

[7.42] Let us then draw lines EA, AH, GA, and GH, and let GH extend in a straight line to K. Hence, the two angles at A [i.e., HAG and GAE] will be equal [by construction], and K [where cathetus HGK and line of reflection EA intersect] will be the image of H.

[7.43] Let arc BD = arc DA, let us draw lines EB, BZ, and BG, and let us extend ZG to L. Thus, the two angles at B [i.e., ZBG and GBE] will be equal, and [the form of] Z will be perceived by the center of sight [according to reflection] from B, and L will be the image of Z.

[7.44] Let us connect KL. KL will therefore be the cross-section of the image of ZH, and since [object-line] ZTH is perpendicular to DE [by construction], and since ZT = TH [also by construction], the two lines EA and AH will be equal [respectively] to the two lines EB and BZ, the two angles [HAG and GAE] at A are equal to the two angles [ZBG and GBE] at B, and line GH = line ZG.

[7.45] Therefore, the two lines AG and GH are equal [respectively] to the two lines BG and GZ, and base AH [in triangle AGH] = base BZ [in triangle BGZ]. Consequently, angle AHG = angle BZL, and angle HAK = angle ZBL. Hence, HK = ZL, and line HG = [line] ZG, so [remainder] GK [of line HK] = [remainder] GL [of line ZL, from which it follows that the two triangles HGZ and GLK are similar and isosceles]. KL is thus parallel to ZH.

[7.46] Moreover, angle HGA is obtuse, and the two angles [HAG and GAE] at A are equal [by construction], so line GH > line GK, and likewise ZG > GL.[103] Hence, line KL < [line] ZH [because of the similarity of isosceles triangles HGZ and GLK]. But KL is the cross-section of the image of ZH. Therefore, line ZH will appear shorter than it actually is. Moreover, line ZH is [on] the viewer's face [insofar as it is a cross-section of the eye that faces the mirror].

[7.47] Therefore, if we rotate the circle at B [i.e., arc BDA] around ED, leaving EG[D] stationary [as the axis of rotation], it will produce a circle, and it will produce a circle on the mirror's surface from the two points A and B. In addition, the position of center of sight E with respect to any line equivalent to ZH on that circle marked off by ZH and with respect to any arc equivalent to arc AB on the segment [of the circle] that the two points A and B mark off on that circle will be equivalent to the position that center of sight E has with respect to line ZH and arc AB. And the proof will be the same whether we suppose the [object-]line to be longer or shorter than ZH.

[7.48] From all of these conclusions, it is clear that the cross-section of the surface of the viewer's face is perceived [to be] smaller than it [actually] is in the concave [spherical] mirror. So it follows that, if the center of sight lies at E, the viewer will perceive his face in such a mirror as smaller than it is, and since K is the image of H, while L is the image of Z, the image will be inverted.

[7.49] Accordingly, the center of sight at E will perceive the viewer's form as such, i.e., it will perceive what lies to the right to the left, and [what lies] below above, and vice-versa. By the same token, if the center of sight lies at any point such that the center of [the mirror's] curvature lies between it and the mirror's surface, it will perceive its [viewer's] form inverted, and this is what we wanted [to demonstrate].[104]

[7.50] It is therefore evident from these four propositions [i.e., 25-28] that in a concave [spherical] mirror [an object] is sometimes perceived as larger, sometimes smaller, and sometimes the same size [as the object itself], and [it] sometimes [appears] properly oriented, sometimes inverted.

[7.51] Moreover, in chapter [2, book 5] on image [formation], we explained that in a concave [spherical] mirror the image will sometimes be single, sometimes double, sometimes triple, and sometime quadruple, and this same phenomenon occurs in the situations just discussed.

[7.52] Hence, whatever yields an image that is larger than itself may yield others that are smaller or the same size, whereas whatever yields a smaller image may yield others larger or the same size, and whatever yields an image the same size [as itself may yield] a larger or smaller [one], and whatever appears upright [according to one image] may appear inverted

according to another image, and vice-versa. So it remains to analyze the forms of those things that are perceived in these sorts of mirrors.

[7.53] **[PROPOSITION 29]** Accordingly, let AB [in figure 6.7.29, p. 136] be a [concave] spherical mirror, let us produce a plane bisecting that mirror through the center, and let it form [great] circle AB centered on E. In this circle let us draw two intersecting diameters, AEO and BED, and let the mirror not extend past arc BADO. Let us then select point Z at random on BE, let us select point K on line AE, and let AK > KE. Then let us connect ZK, and let it continue to F. Let us also draw EF, and let angle EFG = angle EFZ.

[7.54] Thus, since FK > KA, and since KA > KE [by construction], FK > KE.[105] Angle FEK is therefore greater than angle EFK [by Euclid, I.19], so it is greater than angle EFG. Hence, line FG will intersect line KE.[106] Let them intersect at G, then. Consequently, the two lines ZF and FG are reflected at equal angles [ZFE and GFE], so K [where cathetus GEK and line of reflection ZKF intersect] is the image of G if the center of sight is at Z.

[7.55] Now let us draw line ZLH at random, let us connect EH, HG, and ZG, and let us extend FE to M [on GZ]. Accordingly, ZM:MG = ZF:FG [by Euclid, VI.3, since FM bisects angle ZFG]. Furthermore, ZH > ZF [by Euclid, III.7], and GH < GF. Hence, ZH:GH > ZF:FG, so [ZH:GH] > ZM:MG [which = ZF:FG]. Consequently, the line that bisects angle ZHG intersects line MG, so it [also] intersects line EG. Therefore, angle GHE > angle EHZ.

[7.56] Let us take angle EHR = angle EHZ. Line HR therefore intersects line GF, and it [also] intersects line EG, so let it intersect line EG at R. Hence, the two lines ZH and HR are reflected at equal angles [ZHE and RHE], and L [where cathetus REL intersects line of reflection ZLH] will be the image of R [for center of sight Z]. I say, then, that the form of any point on line GR is reflected to the center of sight Z from a point on arc FH, and from no other [arc].

[7.57] The proof of this is [based on] both figures 27 and 28 in chapter [2] on image [formation] in book 5, where it has been shown that the two arcs AB and DO cannot be such that anything on line EO will be reflected from them to [center of sight] Z, and the mirror does not extend to arc BO.[107] Consequently, only arc AD is left [for the reflection].

[7.58] However, in the thirty-fifth proposition [of book 5] it has been shown that the form of any point on diameter EO is reflected at some point on arc AD, and in the thirty-sixth [proposition] of chapter [2, book 5] on image [formation] it was demonstrated that the form of point R is reflected to Z from only one point on arc AD.[108] Therefore, the form of any point on line GR is reflected to Z from one point only on arc AD.

[7.59] Let us take point C on line GR [in figure 6.7.29a, p. 136]. The form of C is therefore reflected to Z from one point on arc AD. I say, then, that that point will lie only on arc FH. For if such is not the case, let the form of C be reflected to Z from U, which lies on arc AF, and let us connect lines ZU, CU, GU, and EU.

[7.60] Therefore, line GU > line GF, and ZU < ZF, so GU:ZU > GF:FZ. Hence, [GU:ZU] > GM:MZ [which = GF:FZ, by previous conclusions]. The line that bisects angle GUZ therefore intersects line ZM, so it [also] intersects ZE. Consequently, angle GUE < angle EUZ; so *a fortiore* angle CUE < angle EUZ, and the same holds for any [other] point on arc AU[F]. The form of C is therefore reflected to Z from arc [D]HF only.

[7.61] I say, furthermore, that it cannot be reflected [to Z] from arc HD. For if that were possible, let it be reflected from Q, which lies on arc HD [in figure 6.7.29b, p. 137], and let us connect lines ZQ, CQ, RQ, ZR, and EQ, and let us extend EH to N. Therefore, line ZQ > [line] ZH, and line QR < [line] HR, so ZQ:QR > ZH:HR, which = ZN:NR [by Euclid, VI.3, since angles ZHE and RHE were constructed equal]. The line that bisects angle ZQR therefore intersects line NR, so it intersects line ER. Consequently angle RQE > angle EQZ, so *a fortiore* angle EQC > angle EQZ. The same result follows for any [other] point on arc HD, so the form of C is not reflected to Z from arc HD or from arc AF.

[7.62] However, it has already been shown that it absolutely must be reflected from arc AD. Consequently, the form of C is only reflected to Z from some point on arc FH. Accordingly, let it be reflected from T [in figure 6.7.29c, p. 137], and let us connect lines CT, ET, and ZT. Therefore, since T lies between the two points F and H, line ZT will lie between the two lines Z[K]F and Z[L]H. Line ZT therefore intersects line KL. Let it intersect it at I, then. I is therefore the image of C, and C has no image other than I.

[7.63] And it will be demonstrated in this way that the image of any point on line GR will be a point on line KL. Thus, KL is the image of [the entire line] GR, and KL is a straight line because it is a segment of the circle's diameter. GR is also a straight line because it too is a segment of the circle's diameter. Thus, in [concave] spherical mirror AB, Z perceives the form of GR according to its proper [left-to-right] orientation, and this is what we wanted [to prove].

[7.64] **[PROPOSITION 30]** Now let us copy the [previous] figure, and let us circumscribe two random arcs on both sides of line GR, namely, GNR and GQR [in figure 6.7.30, p. 138], and let arc GNR not intersect line GH. Let us select point M at random on line GR. The form of M is therefore reflected to Z from [some] point on arc FH. Let it be reflected accordingly from T, and let us connect lines ZT and MT.

[7.65] Hence, the two angles ZTE and ETM are equal, so line MT will intersect arc GNR. Let it intersect that arc at N, then, and let us extend line TM on the side of M. It will therefore intersect arc GQR, so let it intersect at point Q. Next let us connect NE and extend it in a straight line. It will therefore intersect ZT below line KL. So let it intersect that [line] at I. Then let us connect QE and extend it in a straight line. Accordingly, it will intersect ZT above KL. Let it intersect that line at C, then.

[7.66] Consequently, since the two angles [ZITE and NTE] at T are equal, I will be the image of N, and the two points K and L are the images of the two points G and R. The image of arc GNR is therefore a line passing through points K, I, and L, i.e., line KIL. But line KIL is convex with respect to the eye, and arc GNR is convex with respect to the mirror. Z will therefore perceive the form of convex line GNR [as] a convex line.

[7.67] Moreover, since the two angles [ZCTE and QTE] at T are equal, C will also be the image of Q, and line LCK, which is concave with respect to the eye, will be the image of arc GQR, which is concave with respect to the mirror's surface. Z will therefore perceive the form of concave arc GQR [as] a concave line.

[7.68] Thus, in concave [spherical] mirrors a convex line is perceived [as] convex, and a concave [line as] concave in various situations.

[7.69] **[PROPOSITION 31]** Now let there be a concave [spherical] mirror containing great circle ABD [figure 6.7.31, p. 138], let G be the center, let us draw line BG at random, and let us cut from it line GT longer than its half. Let us then draw line ETZ from T orthogonal [to BG], and let both ET and TZ be equal to TG. Let us connect ET, EG, and GZ.

[7.70] Let us then circumscribe a circle around triangle EGZ. It will therefore intersect circle AB at two points, for point T is the center of this [new] circle, and TG > TB [by construction]. So let this circle intersect circle AB at the two points A and D, and let us connect lines GA, GD, EA, EB, ED, ZA, ZB, and ZD.

[7.71] Accordingly, since the two lines ET and TZ are equal [because they are both equal to TG, by construction], the two lines EB and BZ will be reflected at equal angles [i.e., EBG and GBZ subtended by equal arcs]. Also, since the two arcs EG and GZ are equal, the two lines EA and AZ are reflected at equal angles [EAG and GAZ subtended by those equal arcs], and the two lines ED and DZ will [also] be reflected at equal angles [EDG and GDZ subtended by equal arcs].

[7.72] Since GT > TB [by construction], GE > EB. Hence, angle EBG > angle EGB, and angle EGB is half a right angle. The two angles EGB and EBG thus sum up to more than a right angle. Consequently, angle BEG < a right angle, whereas angle EGZ is a right angle. The two lines EB and GZ

will therefore intersect outside the circle [EGZ] on the side of BZ. So let them intersect at M.

[7.73] In addition, since ED lies within angle MEG, it will intersect line GM. Let them intersect at L. And since GB passes through the center of circle EGZ, segment AEG < a semicircle. Therefore, angle AEG is obtuse, whereas angle EGZ is right. The two lines AE and ZG will thus intersect on the side of EG. Let them intersect at F, then. Accordingly, if the center of sight is at E, and if Z lies on some visible object, points M, L, and F will be images of Z. Z is thus perceived at three places.

[7.74] To continue, let us draw a line at random from E to arc DZ, and let it be EK [see inset to figure 6.7.31]. Let us connect GK, let it intersect arc DZ at K, and let us connect lines KZ and GK. Therefore, since arcs EG and GZ are equal, the two angles EKG and GKZ will be equal. [Let us extend GK to point K′ on the mirror, and let us connect EK′ and ZK′]. Consequently, angle EK′G > angle GK′Z.[109] Let angle GK′N = angle EK′G, then. Consequently, the two lines EK′ and K′N will be reflected at equal angles. Let us then extend EK′ to Q. Q will therefore be the image of N with respect to [center of sight] E.

[7.75] Let us now imagine a plane passing along line MGF perpendicular to circle ABD [as represented in figure 6.7.31a, p. 139], let us draw a line from Z in this plane perpendicular to GZ, and let it extend on both sides [of Z]. Accordingly, let it be CZR. Let us then take G as a centerpoint, and let us produce arc CNR of a circle with radius GN. It will therefore intersect line CR at two points, and let them be C and R. Let us connect lines GC and GR. They will thus lie in a plane perpendicular to plane ABG [of the mirror]. Let us then extend GC and GR in a straight line, and at point G let us produce the arc of a circle with radius GQ. It will thus intersect the two lines GC and GR. Let it intersect [them] at S and O.

[7.76] Hence, since the plane of circle ABD [on the mirror] is perpendicular to the plane of the two lines GC and GR, the two angles EGS and EGO will be right. Both planes EGS and EGO will thus be perpendicular to plane SGO, and both of those planes cut a great circle on the mirror that is equivalent to circle ABD. From the counterpart of point K′ in the [great] circle formed by plane EGC, then, two lines between points E and C are reflected at equal angles.[110]

[7.77] Moreover, lines GC and GR are equal, lines GS, GQ, and GO are equal, and Q is the image of N, S is the image of C, and O is the image of R. Therefore, the image of arc CNR, which is convex with respect to the mirror, is arc SQO, which is concave with respect to the center of sight.

[7.78] Meantime, L is the image of Z, and the two points S and O are the images of C and R. Consequently, the image of straight line CZR is a line

passing through points S, L, and O, and such a line is concave with respect to the center of sight.

[7.79] Let us now draw the line passing through points S, L, and O, and let us extend line EG to H. Accordingly, if the mirror does not reach the two points B and H, but one of its two limits lies between the two points B and D, while the other lies inside of H [i.e., between H and D], and if the center of sight lies at E, while the two lines RZC and RNC lie on some visible object, the form of straight line RZC will be concave, i.e., SLO, and the form of convex arc RNC will also be a concave line, i.e., SQO. Furthermore, straight line RCZ will have a single image, and arc RNC will [also] have a single image.

[7.80] Now let us extend BG to I, and let us connect lines EI and IZ. Those two lines will therefore be reflected at equal angles, and EI will intersect FG, so let it intersect at T′. Hence, T′ will be the image of Z. Points M, L, T′, and F will therefore [all] be images of Z. And if the mirror extends beyond the two points A and I, while the center of sight lies at E, and if the viewer faces the mirror on the side of arc AI, he will perceive the entire arc IDA.

[7.81] Consequently, Z will appear at four places, i.e., at L, M, T′, and F, and he will see the two points R and C at the two points S and O, so line RZC will have four concave images. One passes through points S, M, and O, i.e., line SMO; a second will pass through points S, L, and O, i.e., line SLO; a third will pass through points S, T′, and O, i.e., line ST′O; and a fourth will pass through points S, F, and O, i.e., line SFO.

[7.82] From this proposition it is therefore clear that in concave [spherical] mirrors a straight line is perceived as concave, a convex [one] is also perceived as concave, and a straight [line] has several concave forms.[111]

[7.83] **[PROPOSITION 32]** Now let there be a concave [spherical] mirror through whose center a plane passes, let it produce [great] circle ABG [in figure 6.7.32, p. 140], and let D be its center. From D let us draw a line at random, let it be DG, and let it extend beyond the circle. From point D let us extend a line in the plane of the circle perpendicular to line DG, and let it be DA. Let us then cut from right angle ADG a small sub-angle GDE at random such that the difference between right angle [ADG] and angle ADE consists of several [increments of] angle EDG,[112] and let us bisect angle ADE with line DB. Let us also cut [from angle ADG] a sub-angle [ADZ] equal to angle EDG, and let us extend a line from D that forms a right angle with DB [i.e., BDT], and let it be DT.

[7.84] Now let us extend AD on the side of D, let it form DK, and let us extend a line [ZH] from Z that forms with ZD an angle [DZH] equal to

angle KDT. This line will therefore intersect DA because the two angles KDT [which = DZH, by construction] and ADZ sum up to less than two right angles. So let them intersect at H. Angle ZHD is thus equal to angle ZDT.[113]

[7.85] Then from Z let us extend line ZL to form an angle [HZL] with ZH equal to obtuse angle BDK. Accordingly, the two angles LZD and BDZ sum up to less than two right angles, so line ZL will intersect [line] DB.[114] Let it therefore intersect at L.

[7.86] Let us then connect LH, and let us form circle DHL around triangle HLD. It will therefore pass through Z because the two angles LZH and LDH sum up to two right [angles, by Euclid, III.22]. Consequently angles LHZ and LDZ are equal because they are subtended by the same arc [LZ]. But angle ZHD = angle ZDT [by previous conclusions], so it follows that angle LHD = angle LDT. But angle LDT is right [by construction], so angle LHD is right.

[7.87] Now from line DE let us cut line DM equal to [line] DH, and let us connect LM. Angle LMD is therefore right, so circle LHD passes through M and cuts arc HE at a point equivalent to Z.[115] Accordingly, let it intersect at F, and let us draw DF. Angle LDF will therefore be equal to angle LDZ because arc LM = arc LH [by construction according to the bisection of angle ADE by DB], and arc MF = arc ZH [so arc FL subtending angle FDL = arc LZ subtending angle LDZ]. Consequently, arc FMD = arc ZHD.

[7.88] Let us now draw lines HB, HF, FM, FZ, and FB. Angle BHD will thus be acute, while angle GDH will be right [by construction]. Therefore, line HB will intersect line DG outside the circle. Let them intersect at Q, then. HF will therefore also intersect DG outside the circle, so let them intersect at N.

[7.89] Let us then extend FB until it intersects arc LZ. Accordingly, let it intersect at R, and let us connect RM. Angle FRM, which lies on the circumference [of circle ZDF], is thus subtended by arc FM, and angle FBM > angle FRM, but angle FBM lies on the circumference of [circle] ABG. Therefore, if line BM is extended on the side of M, it will cut a larger arc on circle ABG than the counterpart FM [it cuts on circle ZDF].

[7.90] But arc FM [in circle ZDF] is twice its counterpart FE [in circle ABG].[116] Moreover, arc FE = arc ZA [because arcs FM and ZH are equal], whereas arc ZA = arc EG [by construction], and [so] arc FE = arc EG. Consequently, arc GF is twice arc GE, so arc GF [in circle ABG] is the equivalent [in degrees of arc] of arc FM [in circle FDZ].

[7.91] Hence, if BM is extended in a straight line on the side of M, it will cut an arc on circle ABG beyond point F that is greater than arc FG. Line BM will therefore intersect line DG between the two points G and D. Accordingly, let it intersect at O. Let us then extend line FM, and let it

intersect DO at U; let us also extend BM on the side of B, and let it intersect arc LR at C. Let us then connect CD.

[7.92] Accordingly, because angle BFZ lies on the circumference of [circle] ABG, angle BFZ will be half of angle BDZ [by Euclid, VI.33]. But angle BDZ is several times larger than angle ZDA [by construction], so angle R[B]FZ is several times larger than angle ZDH[A]. Consequently, arc RZ is several times the size of arc ZH, and arc CZ > arc RZ, so arc CZ is several times the size of arc ZH.

[7.93] Let us now connect CH. Accordingly, angle CHD + angle C[B]MD = two right angles [by Euclid, III.22], so angle CHD = angle BME [adjacent to BMD]. Therefore, angle ZHD = angle CHD + angle CHZ, which = angle CDZ [since they are both subtended by arc CZ in circle ZDF], and angle CDZ is several times larger than angle ZDA. Therefore, angle CHZ is several times larger than angle EDG [which = angle ZDA, by construction], so angle ZHD exceeds angle CHD by several [increments of] angle EDG. Hence, angle ZHD = angle FMD because arc FMD = arc ZHD [by previous conclusions].

[7.94] Moreover, as we demonstrated [just above], angle CHD = angle BME. Angle FMD thus exceeds angle BME by several [increments of] angle EDG. Therefore, angle FMD exceeds angle OMD by several [increments of] angle EDG [because angle OMD = vertical angle BME]. But angle MOG exceeds angle OMD by [one increment of] angle EDG [by Euclid, I.32], so angle FMD exceeds angle MOG by several [increments of] angle EDG.

[7.95] In addition, angle FMD exceeds angle MUD by only [one increment of] angle EDG [by Euclid, I.32]. Therefore, angle MUD > angle MOG, so angle MOU [adjacent to MOG] > angle MUO [adjacent to MUD]. Hence, line MU > line MO [since MU subtends the larger angle]. And since arc ZHD = arc FMD [by previous conclusions], the two angles HFD and MFD will be equal. The two lines HF and FU will therefore reflect at equal [angles], and likewise HB and BO will reflect at equal [angles]. Consequently, Q is the image of O, and N is the image of U [from the perspective of H, as the center of sight].

[7.96] From M let us then extend a line parallel to line HQ, let it be MS, and let us also extend a line from M parallel to line HN, and let it be MP. Thus, since angle HND > angle HQD, angle MPO > angle MSO. P therefore lies between the two points S and U, and because angle HDN is right [by construction], angle HND will be acute. Accordingly, angle MPD is acute, so angle MPS [adjacent to it] is obtuse. Line MS is therefore longer than [line] MP.

[7.97] But MU > MO, as we [just] established, so SM:MO > PM:MU, and SM:MO = QB:BO because MS is parallel to BQ. Likewise, PM:MU =

NF:FU, so QB:BO > NF:FU. Moreover, QB:BO = QD:DO, whereas NF:FU = ND:DU, as we showed in the twenty-fifth proposition of chapter [2, book 5] on image [formation].[117] Therefore, QD:DO > ND:DU.

[7.98] Now that these points have been established, let us redraw the circle and finish the proof so as to avoid adding lines and confusing letter-designations. Accordingly, let ABG [in figure 6.7.32b, p. 142] be the circle in the second version [of figure 6.7.32, p. 140], let D be its center, and let us draw line DQ. Let DU [in the second version] be equivalent to DU in the original version, and let DO [in the second version] be equivalent to DO in the original version. Also, let DQ [in the second version] be equivalent to its counterpart in the original version, and likewise for DN.

[7.99] Let us then draw DH′ perpendicular to DQ [as well as] to the plane of the circle, and let DH′ be equivalent to its counterpart [DH] in the original version. Angle H′DQ will therefore be right, and the [great] circle that [the plane containing] H′DQ produces in the mirror will be among the circles [like ABG] within which a form is reflected. Furthermore, the arc that lines H′D and DQ measure off [in the great circle produced on the mirror by plane H′DQ] will be equal to arc AG in the original circle. And from the two points on this [arc] that are equivalent to the two points B and F [on arc AB in the original circle] lines from the two points U and O will be reflected at equal angles [to point H′]. Q will therefore be the image of O [for center of sight H′], and N [will be] the image of U.[118]

[7.100] From U let us produce a line perpendicular to line DU within the plane of circle ABG, and let it be ZUE. Let D be the center, and [from it] let us produce the arc of a circle with a radius of DO. It will therefore intersect line ZUE at two points. Accordingly, let it intersect at Z and E, and let it form arc ZOE. Let us then draw DZ and DE, and extend them beyond the circle. At a radius of DQ, let us produce arc TQK around D. It will therefore intersect the [extension of the] two lines DZ and DE at T and K. Let us then draw TK. Accordingly, it will intersect line DQ at L.

[7.101] Consequently, since H′D is perpendicular to the plane of the circle, both angles H′DT and H′DK will be right. Moreover, both planes H′DT and H′DK produce a [great] circle on the mirror's surface, and the arc on it that lies between the two lines H′D and DT will be equal to the arc lying between HD and DQ [in the original figure—i.e., 6.7.32], and the same holds for the arc between the two lines H′D and DK. In addition, both lines DZ and DE are equal to line DO. These two arcs [cut on the mirror's surface by planes H′DZ and H′DE] are therefore of the kind from which lines will be reflected at equal angles to the two points Z and E.[119] Furthermore, the two lines DT and DK are equal to line DQ, so point T is the image of Z, and [point] K is the image of E.

[7.102] Since, moreover, lines DT, DQ, and DK are equal, and since lines DZ, DO, and DE are equal, DT:DZ = QD:DO = KD:DE. But, as we showed in the previous theorem [i.e., in paragraph 7.97 keyed to figure 6.7.32], QD: DO > ND:DU. Therefore, DT:DZ > ND:DU, and the same holds for KD: DE.

[7.103] In addition, since the two lines ZD and DE are equal, and since the two lines DT and DK are equal, line TK will be parallel to line ZE. Therefore, DT:DZ and KD:DE will [both] be as LD:DU. Hence, LD:DU > ND:DU, so line LD > line ND. N therefore lies between L and U. But N is the image of U, and the two points T and K are the images of Z and E. As a result, the image of straight line ZUE is the line that passes through points T, N, and K. But the line that passes through these points is convex, from which it is clear that in concave [spherical] mirrors a straight line sometimes appears convex in certain situations.

[7.104] Now let us take some point M at random on line ZU [in figure 6.7.32d, p. 143], and around M as center let us produce arc RUF with radius MU. This arc will therefore intersect arc ZOE[120] in two points. Accordingly, let it intersect at R and F, let us draw lines DR and DF, and let them pass in a straight line until they intersect arc TQK at C and I. The plane containing the two lines H′D and DC will therefore produce a [great] circle on the mirror from whose circumference lines from R will be reflected at equal angles, and by the same token the plane containing the two lines H′D and DI will produce a [great] circle on the mirror from whose circumference lines will be reflected to F [at equal angles]. C is therefore the image of R, I is the image of F, and N is the image of U.[121]

[7.105] Consequently, the image of arc RUF is the line passing through C, N, and I, but this line will be convex, whereas arc RUF is concave with respect to the mirror's surface. Therefore, when the center of sight is at H′, and when any of the lines ZUE, ZOE, or RUF lies on some visible object, straight line ZUE will be perceived [as] convex, convex line ZOE will be perceived [as] concave, and concave [line RUF is perceived as] convex. Consequently, if each of the lines ZUE, ZOE, and RUF has [only] one image, the form of those lines will be just as we showed. But if they have additional images, they may be similar to the other images [i.e., the ones just discussed], or they may be different.

[7.106] From these propositions [i.e., 29-32] it is therefore evident that straight lines are sometimes perceived [as] straight in concave [spherical] mirrors, sometimes [as] convex, and sometimes [as] concave. In addition convex lines are sometimes perceived [as] convex [and] sometimes [as] concave, and concave [lines] are sometimes perceived [as] convex [and] sometimes [as] concave.

[7.107] So the forms of visible surfaces are perceived [as] other than they [actually] are in these sorts of mirrors, for straight lines exist only in flat surfaces, and when a straight line that exists in a plane surface is perceived [as] convex or concave, the surface in which it lies will be perceived [as] convex or concave. Accordingly, when the eye perceives convex, concave, and straight lines other than they [actually] are, it will perceive the surfaces in which they lie other than they are.

[7.108] From the foregoing, then, it is clear that in everything that is perceived in concave [spherical] mirrors, a misperception occurs, but in certain cases it occurs in every situation, without exception, whereas in certain [cases] it occurs in a specific situation. Moreover, compound misperceptions arise in these mirrors just as in the case of compound illusions [in the other mirrors],[122] and this [is what] what we wanted to demonstrate.

CHAPTER EIGHT
Concerning Misperceptions That Arise in Concave Cylindrical Mirrors

[8.1] In these [sorts of mirrors] the same things happen as happen in concave spherical [mirrors], for the misperceptions that arise from reflection [by itself] occur [in these mirrors], i.e., the weakening of light and color and the variation in situation and distance that occur in all mirrors. Moreover, variation in size occurs in these mirrors in the same way as it happens in concave spherical mirrors. Also, one visible object appears [as] one, or [as] two, or [as] three, or [as] four, and [it appears] properly oriented or reversed in various circumstances, and a flat object appears concave or convex. So let us show how the size and number of a visible object may vary in these mirrors, as well as how it appears properly oriented or reversed in the way that we demonstrated [these phenomena] in spherical concave mirrors.

[8.2] **[PROPOSITION 33]** Let us recapitulate the first of the two diagrams provided in [the analysis of] misperceptions [occurring in] convex cylindrical mirrors [i.e., figure 6.5.18, p. 127, redrawn as figure 6.8.33, p. 144], and [let us use] the same letters. Now in that proposition [i.e., proposition 18, in conjunction with proposition 17, pp. 190-193 above] it was shown: that lines EG and GT, EB and QB, and EA and AH are reflected at equal angles; that lines EO, HA, BQ, and TG intersect at O;[123] that line ABG is a straight line extended along the longitude of the mirror; that lines GZ, BL, and AD are perpendicular to the plane tangent

to the [mirror's] surface and passing along line ABG; that line ABG is perpendicular to the plane containing triangle EBO; that line TQ = [line] QH, and [line] AB = [line] BG; that S, C, and I are the images of H, Q, and T; that C lies nearer point E than [straight] line SI; that [straight] line SI lies in the plane of triangle UHT; that the two lines UH and UT are equal; that the two lines US and UI are equal; and that the two lines ES and EI are equal.

[8.3] Let us draw CU, and let it intersect SI at F. It will therefore bisect this line [i.e., SI] because HT is bisected at Q [by construction], and CU will lie in the plane of triangle CUE, which lies in the plane of the circle [passing through] B parallel to the base of the mirror.[124] Q will therefore lie in the plane of triangle CUE, and C lies in triangle CEI. Hence, C lies on the common section of these two planes. But this [common] section is line EB; so C lies on a straight line [with] EB.[125]

[8.4] Moreover, the two lines HU and TU [in figure 6.8.33] lie outside the two points D and Z [on the axis], for the two lines HU and TU are the normals passing from H and T to the two lines that are tangent to two sections [on the cylinder's surface] on whose periphery points A and G lie [i.e., the two elliptical sections formed on the mirror's surfaces by planes of reflection TGE and HAE]. Consequently, the plane of triangle UHT lies outside axis DLZ.

[8.5] Even if the axis is extended to infinity, however, no point on it will lie in the plane of triangle UHT, for if it did, then if it were to be connected in a straight line with some point on line HT, the plane in which that straight line and line HT lay would be the plane of triangle UHT, and that plane would be the one in which the two parallel lines HT and DZ lie. Hence, the plane containing the two lines HT and DZ [supposedly] forms the plane of triangle HUT, so the axis will lie in the plane of triangle HUT.

[8.6] But the axis is parallel to line HT by construction, and the axis [supposedly] intersects the two lines HU and TU. Moreover, line TH lies in [i.e., passes through] the plane of triangle UEH, which is the plane of reflection [for object-point H and reflection-point A], and the common section of this plane and the surface of the mirror is some [elliptical] section. Therefore, plane EUH intersects the axis of the cylinder in one point, that is, D, as we showed before [in proposition 18]. And if the axis intersects line HU, the point of intersection with line HU will lie in the plane of triangle UEH. But there is no point in this plane other than D through which the axis might pass. Therefore, line HU intersects the axis at D. But it has already been shown that HU intersects it at a point beyond D, which is impossible.[126]

[8.7] Consequently, axis DZ lies outside the plane of UHT and nearer to point E than plane UHT [i.e., between E and plane UHT]. The plane in which lines HT and DZ lie is therefore nearer to point E than plane UHT.

Moreover, C lies in the same plane as HT and DZ because it lies on line QL, and QL lies in the same plane as HT and DZ. Therefore, C lies nearer to point E than [do] S and I. But C lies in a straight line with EB. If, therefore, EB is extended toward B, it will reach C, so let it reach C.

[8.8] Now that these points are established, I say that if line SI, which is parallel to the mirror's axis, lies on some visible object, if the center of sight lies at O on the concave side of the cylinder, and if the reflecting surface is a concave surface, then SI will be perceived by O in concave mirror ABG, and its images will vary according to how its distance from the axis varies.

[8.9] The proof of this [claim] lies in the fact that angle EBM is acute, so [vertical] angle LBC is acute. Moreover, line EBC lies in the plane of the circle [passing through] B, and LB is [on] the diameter of this circle. Hence, EB intersects the circle, so CB lies inside the mirror's concavity.

[8.10] By the same token, OB lies inside the mirror's concavity because angle OBL is acute, and the two angles OBL and CBL are equal, since they are equal to the two angles EBM and QBM, while LB is perpendicular to the plane that passes through B tangent to the cylinder. The form of C thus passes along CB and reaches B, and it is reflected along BO and perceived by the center of sight at O.

[8.11] Furthermore, when we discussed convex cylindrical mirrors in chapter 5 [of book 4, paragraph 5.18, in Smith, *Alhacen on the Principles*, 332], we showed that the plane tangent to the cylinder at G will lie below E. Therefore, EG intersects the tangent plane, so it intersects the line tangent to G on the periphery of the [elliptical] section [formed on the mirror by the plane of reflection]. As a result, it intersects the [elliptical] section and falls inside it; so it will fall inside the concavity of the mirror. The two lines OG and GI thus lie inside the concavity of the mirror, whereas ZG is perpendicular to the plane tangent to the cylinder at G, and the two angles OGZ and IGZ are equal. Hence, the form of I passes along IG, reaches G, is reflected along GO, and is perceived at O along line GO. So too, [the form of] S passes along SA and is reflected along AO.

[8.12] But when we dealt with misperceptions [arising] from convex cylindrical mirrors, we demonstrated [in proposition 16, lemma 5] that the two lines HU and TU are normal to the two planes tangent to the [elliptical] sections passing through the two points A and G. Therefore, the image of S lies on line HU. Moreover, OA is the radial line extending from the center of sight to the point of reflection, so the image of S lies on OA. H [where the two lines intersect] is therefore the image of S, and it is shown in this way that T is the image of I.

[8.13] Let us then connect CL. Accordingly, since [the form of] C is reflected to O from the periphery [of the circle passing through] B, [its] image Q will lie on line CL. And OB is the radial line extending between

the center of sight and the point of reflection, so the image of C lies on line OB. Consequently, the image of C lies at the intersection of [cathetus] QL and [line of reflection] OB [i.e., at Q].

[8.14] When we dealt with images in concave spherical mirrors in chapter [2, book 5] on image [formation], however, it was shown [in proposition 32, in Smith, *Alhacen on the Principles*, 446-448] that the image of a point whose form is reflected from the concavity of a [great] circle [on the mirror's surface] may intersect the radial line linking the center of sight and the point of reflection beyond the circle, or between the center of sight and the circle, or at the center of sight [itself], or behind the center of sight, or CL may be parallel to OB.

[8.15] In that [same] chapter [on image formation], moreover, it was shown that the image might consist of a single point, or of two, or of three, or of four.[127] So the image of C might lie [at some point between B and Q] on BQ, or perhaps beyond Q, or perhaps on BO, or perhaps at O, or perhaps behind O.[128] Moreover, the image of C might consist of a single point, or of two, or of three, or [of] four.

[8.16] Accordingly, if the image of C lies at Q, then HT will be the cross-section of SI's image. So, if all the images of [points on] SI lie on line HT, its form will be a straight line. If not, however, it will be nearly straight because its midpoint lies on a straight line between two endpoints.[129] Nevertheless, if the image of C lies beyond Q, the image of SI will be somewhat concave with respect to the center of sight. And if the image of the visible [point C] lies on line BO [i.e., in front of Q], then the image of SI will be convex with respect to the center of sight.

[8.17] Moreover, if the image of C consists of several points, the image of C will lie on several lines, all of whose endpoints converge at the two points H and T, and their midpoints are distinct and separate. In addition, HT forms the cross-section of image SI, no matter how that image is formed, and [this] cross-section is common to all of its images if it has several images, and line HT [on the image] is longer than [line] SI [on the object] by some amount.[130]

[8.18] It is therefore evident that, when straight lines parallel to the axis of a concave cylindrical mirror lie on some visible object, the image of any [one of them] may be straight or concave, and it may consist of a single [line] or [of] several.

[8.19] **[PROPOSITION 34]** Now let us recapitulate the second diagram [provided in the analysis] of misperceptions in convex cylindrical mirrors [i.e., figure 6.5.19, p. 128, which accompanies proposition 19]. In this proposition [represented by figure 6.8.34, p. 147, abstracted from figure 6.5.19], it has been shown: that the two lines EB and BH are reflected at

equal angles; that the two lines EG and GT are reflected at equal angles; that HB and TG converge at L; and [that] HB forms an acute angle with BO. Consequently, HB intersects the plane tangent to the surface of the cylinder at B, so BL lies inside the concavity of the cylinder. And the same holds for GL, as well as for the two lines BR and GY.

[8.20] Moreover, the two angles LBD and DBR are equal, and the two angles LGD and DGY are equal. Hence, if RY lies on some visible object, if the center of sight lies at L, and if the concave surface of the cylinder is polished [and therefore reflective], the form of R is extended along RB and reaches B. It will then be reflected along BL and will reach L, and it will be perceived by L. Moreover, line HU [i.e., the cathetus dropped from R through the mirror's surface] is perpendicular to a line tangent to the [elliptical] section from whose periphery the two lines RB and BL will be reflected. Therefore, H is the image of R. Likewise, it will be proven that the form of Y is extended along YG and is reflected along GL, and its image is T.

[8.21] Let us now draw KU [in figure 6.8.34a, p. 148].[131] Accordingly, it will intersect RY at M. M therefore lies in the plane passing through the axis and through L, for L and K lie in that plane, so KU lies in that plane. Moreover, since the two points M and L lie in a plane passing through the cylinder's axis, the form of M will be reflected to L within that plane. Line AZ is the common section of the cylinder's surface and the plane passing through its axis and through L, so the form of M will be reflected to L from [a point on] AZ.

[8.22] Let us then connect EM, which lies in this plane. EL also lies in this plane, and E lies above the plane tangent to the surface of the cylinder along line AZ. Hence, if AZ is extended in a straight line on the side of Z, it will intersect the two lines EM and EL. Accordingly, let it intersect EM at I and EL at N. N therefore lies between the two points E and L because L lies inside the concavity of the cylinder, whereas N lies on the cylinder's surface, and E lies above the [surface of the] cylinder.

[8.23] Furthermore, in the proof based on this diagram, it was shown that circle BZG lies halfway between [the plane passing through] line HT [parallel to the base of the cylinder] and the plane passing through E parallel to the base of the cylinder. In addition, the perpendicular [EX'] that passes from E through AZ lies in the plane passing through E parallel to the cylinder's base. Therefore, the perpendicular passing through E to line AZN falls outside triangle EIN and on the side of N. Consequently, angle EIN is acute; so [vertical] angle MIA is [also] acute.[132]

[8.24] So let us extend MQ [in figure 6.8.34b, p. 149] from M perpendicular to AI. Q will therefore lie beyond I with respect to N [i.e., below I on AZA']. And let us extend MQ on the side of Q, and let us cut

off QS equal to QM. S will therefore lie beyond the surface of the mirror and outside its concavity, while L will lie inside its concavity.

[8.25] Let us then draw LS. Accordingly, it will intersect NQ at F, and from F let us extend FX parallel to QM. It is therefore perpendicular to AN and [lies] in the plane passing through the axis and through L, so it is a diameter of the circle [produced on the mirror's surface by the plane] passing through F parallel to the cylinder's base. Therefore, line XF is perpendicular to the plane tangent to the cylinder and passing along AZ.

[8.26] Let us connect MF. Accordingly, it will be equal to FS, and [so] the two angles [FMQ and FSQ] at M and S will be equal [because triangles FMQ and FSQ are equal, by Euclid, I.4]. Moreover, because XF is parallel to MS, the two angles [LFX and MFX] at F will be equal to the two angles [FSQ and FMQ] that are at S and M.[133] The two lines MF and FL are therefore reflected at equal angles, and XF is perpendicular to the plane tangent to the mirror's surface at F. So the form of M is extended along MF and is reflected along FL, and its image will be S.

[8.27] Moreover, since the two lines RY and HT are parallel and [therefore] perpendicular to the plane passing through the axis and through L (because HT was posited as such [in proposition 19]), the two planes passing through the two lines HT and RY [perpendicular to the axis] will be parallel. Since, moreover, RY is perpendicular to the plane passing through the axis and through L, the plane [consisting] of the two lines RM and MS will be perpendicular to the plane passing through the axis and through L. Furthermore, MS will be the common section of these two planes [i.e., RMS and ELDS in figure 6.8.34b], and since AQ lies in the plane [ELDS] passing through the axis and is perpendicular to MS, which is the common section of the plane [ELDS] passing through the axis and the plane [consisting] of the two lines RM and MS, AN will be perpendicular to the plane [consisting] of the two lines RM and MS.

[8.28] But line AN is parallel to the axis of the cylinder, so the axis of the cylinder is perpendicular to the plane containing the two lines RM and MS. Therefore, this plane is perpendicular to the axis of the cylinder. S therefore lies in the plane passing through line RY perpendicular to the axis of the cylinder.

[8.29] But line HT lies in a plane perpendicular to the axis of the cylinder and parallel to the plane passing through line RY. Hence, S lies outside [and above] HT and nearer to L than HT. In addition, the two points H and T are the images of R and Y, and point S is the image of M, so the image of line RMY is the line passing through H, S, and T.

[8.30] Such a line is curved, however, because S lies outside HT, and a curved line HST must pass through points H, S, and T. And since HT lay beyond the convex [side of the] cylinder, according to construction, HT

will lie beyond the surface of the mirror with respect to L. In addition, we have already shown that S lies beyond the concavity of the mirror with respect to L, so the entire line HST lies beyond the concavity of the mirror's surface. Moreover, L lies inside the concavity of the mirror, so L lies outside the plane containing line HST. Therefore, the curvature of line HST will appear clearly to the eye at L.

[8.31] Furthermore, since F lies on the surface of the cylinder, while TH lies beyond the cylinder, and since TH lies in the plane of triangle LHT, line LFS will be higher than the plane of triangle LHT. Line LS will therefore be higher than the two lines LH and HT with respect to the center of sight at L. Consequently, S is higher than the two points H and T, so line HST will appear concave to the center of sight at L.

[8.32] **[PROPOSITION 35]** To continue, let us cut the cylinder with a plane slanted to its axis, but do not let it pass through the entire axis [so as to cut the cylinder along a line of longitude]. Accordingly, it will form an [elliptical] section. Let it therefore be ABG [in figure 6.8.35, p. 150].[134] But in the first of the propositions concerning concave cylindrical [mirrors] it was demonstrated that in the plane of any [elliptical] section on a cylinder there will be a normal to the plane tangent to the cylinder from whose endpoints forms are reflected.[135] So let that normal be GZ[A], let BE[K] be perpendicular to the line tangent to the periphery of the [elliptical] section at B, and let B lie near G. BK will therefore intersect normal GZ, and it will form an acute angle with it. Accordingly, let it intersect at E. Angle BEG will therefore be acute [by proposition 16, lemma 5].

[8.33] From G let us extend line GD parallel to line BK. Hence, angle DGE will be acute, so GD will lie inside the concavity of the cylinder. Let us then take angle EGL equal to angle EGD. GL will thus intersect BE at L. And let us mark M on line LE [inside the elliptical section]. Angle MAG will therefore be acute because AM lies inside the [elliptical] section.

[8.34] Let us then take angle GAD equal to angle GAM. Therefore, AD will intersect GD, since the two angles [GAD and AGD] that are at A and G are acute. So let them intersect at D. AD will therefore intersect BK. Let it then intersect at T.

[8.35] Consequently, if BK lies on some visible object, and if the center of sight lies at D, the form of L will be seen at G because the form of L will be reflected to D from G, and DG is parallel to normal LB [by construction].[136] Meantime, the form of M is seen at T because the form of M is reflected to G from A, and T is the image of M.

[8.36] Now let a plane pass through D parallel to the base of the cylinder [to form the circle represented at the bottom of figure 6.8.35]. Accordingly, it will intersect the plane of ABG and will form circle COR on the surface

of the cylinder. The plane of this circle will therefore intersect BK, for it intersects GD, which is parallel to it [by construction]. Let it therefore intersect BK at K, and let point H be the center of circle CR. Let us then draw DH, and let it pass to R. Let us also draw KH, and let it pass to C.

[8.37] Hence, the form of K is reflected to D from the periphery [of the circled centered on H] within arc RC, as was shown in [the analysis of] images [formed] in [great] circles [within concave spherical mirrors].[137] So let it be reflected from O, and let us draw KO, DO, and HO. The angles [DOH and KOH] at O are therefore equal, and [reflected ray] DO will intersect [cathetus K]HC at N. So N is the image of K.

[8.38] Let us then connect KD. Accordingly, KD will be the common section of circle RC and [elliptical] section ABG, since the two points K and D lie in both planes, for there is nothing except line KD in plane ABG of the [elliptical] section that is [also] in the plane of circle RC. G therefore lies outside the circle, and likewise T, and both lie in the plane of the [elliptical] section.

[8.39] N, meanwhile, lies in the plane of the circle, and the form of LMK passes through points G, T, and N, and [so] the line that passes through these points is curved. However, the plane of the [elliptical] section is slanted with respect to the cylinder's surface, so the [major] axis of the [elliptical] section does not pass along the entire axis of the cylinder, nor is it parallel to the cylinder's base.

[8.40] From this and the previous two propositions it is therefore evident that straight lines parallel to the axis of the cylinder, as well as those parallel to its base, and also those that are slanted with respect to its surface may appear curved, or straight, or reversed. Furthermore, since T is the image of M and N the image of K, the form of MK will be reversed.

[8.41] In addition, if the line also lies in the plane of a circle parallel to the cylinder's base, and if the plane of that circle passes through the center of sight, the image may be the same size [as its object] and properly oriented, or it may be reversed, as was claimed in [propositions 25-27 of] the seventh chapter of this book [dealing] with images in [great] circles [on concave spherical mirrors].

[8.42] It is thus evident that the forms of objects perceived in concave cylindrical mirrors may be properly oriented, or they may be reversed.

[8.43] **[PROPOSITION 36]** Now let us recapitulate the diagram for the third proposition [dealing] with misperceptions in concave spherical mirrors, leaving the letters as they are [in figure 6.8.36, p. 151, which combines figures 6.7.26 and 6.7.27]. Let BZA be a circle on the surface of a concave cylindrical mirror, and let the center of sight be at D [on DG, which is perpendicular to the plane of BZA]. It will therefore lie outside

the circle's plane, and the two lines EA and EB will [each] be perpendicular to a plane tangent to the cylinder's surface [at points A and B]. In addition, the plane of triangle DGE will be perpendicular to the plane of the circle because DG is perpendicular to the plane of the circle.

[8.44] Hence, the plane of triangle DGE passes through the entire axis as well as through D, whereas neither plane DBO nor plane DAO, which intersect along line DO, passes through the entire axis. Moreover, there is nothing but E on the cylinder's axis in either plane, E being the circle's center. And each of the planes DBO and DAO forms an [elliptical] section on the cylinder's surface, and forms are reflected from these [elliptical] sections at the two points A and B.

[8.45] The form of R is therefore reflected to D from B, whereas the form of M is reflected to D from A, and NU will be the cross-section of the image of MR, and it is shorter than MR. Likewise, the [forms of the] two points H and L are reflected to D from the two points A and B, and TK will be the cross-section of LH's image, and it is the same size as TK. Finally, CI will be the cross-section of FQ's image, and it is longer than FQ. All of these images, moreover, will be reversed.

[8.46] But if the center of sight lies at O, and if lines CI, TK, and NU are the visible objects, the opposite will obtain, for in that case the cross-section of the image [FQ] of CI will be shorter than CI, whereas the cross-section of the image [MR] of NU will be longer than NU, and the cross-section TK [of LH's image] will be the same size as it, and these images will all be properly oriented. All these points were shown in the preceding chapter.

[8.47] Furthermore, when either endpoint of any of these [lines] has a single image, and when any intermediate point [on that line] has several images, that line will have as many images as the intermediate point has. If, moreover, one endpoint or the other [of the line] has several images, and if the intermediate point has one, then the line will have as many images as the endpoint has. And if one endpoint or the other has several images, and if the intermediate point has several images, the line will yield images according to the greatest number [as pointed out in note 130, p. 256]. This will be shown as was shown for images in concave spherical mirrors.

[8.48] Hence, in concave cylindrical mirrors misperception occurs in all respects as it occurs in concave spherical mirrors, that is, concerning the shapes of visible forms, concerning the sizes and number of their images, and concerning their proper orientation or reversal, along with the misperceptions that apply to reflection [itself]. And the misperceptions in these cases will be as they are in the aforementioned mirrors, and these are the points we wanted to demonstrate in this chapter.

CHAPTER NINE
Concerning the Misperceptions That Occur in Concave Conical Mirrors

[9.1] In these [sorts of mirrors] the misperceptions that occur are those that occur in concave cylindrical mirrors. Indeed, the weakening of color and light, as well as variation in location and distance, occur in these mirrors as in all [other kinds of] mirrors, for the cause of this is reflection [itself]. In addition, a multitude of images arises in these mirrors, just as in concave cylindrical and spherical mirrors, as was claimed in chapter [2, book 5] on image [formation]. What happens in these mirrors is also like what happens in concave cylindrical [mirrors], i.e., what is straight appears convex, or it appears concave.

[9.2] The demonstration of this is that straight lines that extend along the length of the mirror and pass through the cone's vertex, as well as those that are near these [lines in orientation], appear convex, or they appear concave, or perhaps [they appear] straight.

[9.3] **[PROPOSITION 37]** The demonstration of this point is like the demonstration [given] for concave cylindrical mirrors [in proposition 33, pp. 221-224 above], for if we recapitulate the second diagram concerning misperceptions in convex conical mirrors [i.e., the top diagram of figure 6.6.22a, p. 131, from which figure 6.9.37, p. 152, has been abstracted], we will find the cross-section of the image of straight line [AN] placed toward that mirror, which, in that case, is [curved line] A[P]Y inside the concavity of the conical mirror, and we will find the point that is below the plane tangent to the cone and passing along the line [AZE] from which the form of the straight line [AN] is reflected to the center of sight, which is F in that case.

[9.4] If [that] point is the center of sight, all the points that are on the cross-section of the image will be reflected to point F, and the images of the two endpoints A and Y [on object-line APY] will be the endpoints of straight line AN, and the image-location of intermediate [point P] on AY will vary. And this will be demonstrated by the same train [of logic] we followed in the proof [provided] in the first proposition [dealing] with concave cylindrical mirrors [i.e., proposition 33].[138]

[9.5] From this it is clear that, if AY lies on some visible object, and if F is the center of sight, the image may appear convex, or it may appear concave.[139] And it is also evident from the second proposition concerning misperceptions in concave cylindrical mirrors [i.e., proposition 34] that lines

placed along the width of a [concave conical] mirror will appear concave with a remarkable curvature and that images of straight lines that lie in planes passing through the axis and the center of sight will be straight.

[9.6] **[PROPOSITION 38]** Now with the same letters, let us recapitulate the third figure concerning misperceptions in concave spherical mirrors [as given in figure 6.8.36, p. 151, which combines figures 6.7.26 and 6.7.27]. If, therefore, some point [i.e., E] lies on the axis of the cone, and if the two lines EA and EB [passing through that point] are perpendicular to planes tangent to the cone (and this is possible because they are equal, since they can form two equal acute angles with the axis), then when [each of] these two lines is perpendicular [to a plane tangent to the mirror], and when the center of sight is at D, the plane containing lines GE and ED will pass through the entire axis as well as through the center of sight.

[9.7] Furthermore, both planes [containing] DAM and DBR will be inclined to the axis of the cone, and [so] the common sections of those two [planes and the mirror's surface] will be conic sections. The form[s] of points R, H, and Q will be reflected to D from B, and the forms of points L, M, and F are reflected to D from A. Hence, if lines MR, LH, and FQ lie on some visible surface, and if the eye is at D, then NU will be the image of MR, TK will be the image of LH, and CI will be the image of FQ.

[9.8] Thus, the image [NU] of MR will be shorter than [MR] itself, the image [CI] of FQ will be longer than [FQ] itself, and the image [TK] of LH will be the same size as [LH] itself, and all the images will be reversed.

[9.9] If, moreover, the center of sight is at O, and NU, TK, and CI are on the surfaces of visible objects, their images will be MR, LH, and FQ. Accordingly, the image [FQ] of CI will be shorter than [CI] itself, the image [MR] of NU [will be] longer [than NU itself], and the image [LH] of TK will be the same size [as TK itself].

[9.10] And these images will be properly oriented, for these images will lie behind the center of sight and will be perceived facing the center of sight along [direct] radial lines.[140] Consequently, points M, L, and F are perceived along line [of reflection] AO, whereas points R, H, and Q are perceived along [line of reflection] OB, and so their form will be reflected with proper orientation.

[9.11] From what we have claimed in this chapter, therefore, it is clear that straight lines sometimes appear convex in these mirrors, sometimes concave, and sometimes straight, and [they] sometimes [appear] longer, [sometimes] shorter, and [sometimes] the same size [as they actually are], and [they sometimes appear] properly oriented, and [sometimes] reversed.

[9.12] In chapter [2, book 5] on image [formation], moreover, we showed that in mirrors of this kind every visible point sometimes has one image, sometimes two, or three, or four. Therefore, misperception occurs in everything that is perceived in this sort of mirror, just as in concave cylindrical [mirrors], and compound misperceptions also occur in these as in the rest of the mirrors. Examples and proof of these [kinds of compound misperceptions] are as [they can be found] in plane mirrors. And we intended to explain this in this chapter. Now, however, let us end the sixth book.

NOTES TO BOOK SIX

[1]In other words, the object may appear to lie farther behind the mirror than it should because of its diminished brightness or because the ambient light is diminished enough to cause a misjudgment of the distance.

[2]Throughout his analysis of mirror imaging in book 6, Alhacen focuses on facing images, the paradigm case being that in which the viewer sees his own image in a directly facing mirror. Thus, in the case of plane mirrors, the right-hand side of the viewer's face will be seen on the right-hand side of the facing image, and vice-versa for the left-hand side. A more general and "objective" way of understanding image-reversal according to Alhacen's analysis is as follows. Assume that both ZH and its image DG in figure 6.3.1 are facing objects, and suppose that viewpoint E is posed between them. When facing ZH, E will see point H on the right-hand side of the object and point Z on its left-hand side. When facing DG, however, E will see point G—which corresponds to right-hand point H on ZH—on the left-hand side of the object and point D—which corresponds to left-hand point Z on ZH—on the right-hand side of the object. Consequently, the left-right orientation of the image is opposite to that of the object. On the other hand, the up-down orientation remains the same, so there is no accompanying inversion, as there is in the case of concave spherical mirrors.

[3]What Alhacen is getting at here can be illustrated by figure 6.3.1. If ZH is the object, DG its image, and E the center of sight, then, because image DG lies below the mirror, its apparent distance from E will be greater than the actual distance between E and the object ZH. Image DG will therefore appear commensurately smaller than object ZH. In addition, image DG is dimmed by the weakening of its light and color by reflection itself. It will therefore appear even farther away from E than it would if it were an actual object whose light and color were not dimmed by reflection.

[4]This is a misperception of separation or disjunction, and it is akin to the misperception described in 3, 7.77-78 (Smith, *Alhacen's Theory*, 611).

[5]That AD:DT = AM:MT and BD:DQ = BL:LQ follows from the fact that M is the endpoint of tangency for reflection-point H and L the endpoint of tangency for reflection-point N.

[6]FB = FM – BM, and KT = MT – MK. Since it has been established that FM:MT = BM:MK, then, by Euclid V.19, it follows that (FM – BM):(MT – MK) = FM:MT. But FM:MT = BL:LQ by construction. Thus, FB [which = FM – BM]:KT [which = MT – MK] = BL:LQ.

[7]If angle EDH is right, then of course line EH is the hypotenuse, which is the longest of the three sides in the triangle. By the same token, if angle EDH is obtuse, then EH will be the longest side because it will subtend the largest angle in the triangle. Hence, since EH < AB, then *a fortiori* ED < AB.

[8]Note the use of "form" rather than "image" here, a conflation that occurs throughout book 6 and that appears to have no significance beyond a relaxing of

terminological distinctions. The case in which image EH > object AB is dealt with in the very next proposition.

[9]As will be evident later (see note 24 below), this restriction on the length of ZD is arbitrary and pertains only to the construction and analyis from this point to the end of paragraph 4.52.

[10]This is tantamount to finding a third proportional, HT, such that AH:HD = HD:HT, which can be done by Euclid, VI.11.

[11]That CH,HA = $3HD^2$ follows from the fact established earlier that AH:HN = HQ^2 = one-fourth AH,HT. But it was also established earlier that HQ^2 = one-fourth HD^2, so one-fourth AH,HT = one-fourth HD^2. Accordingly, since CH = 3HT, then CH,HT = 3AH,HT = $3HD^2$. From this, of course, it follows that AH,HT = HD^2, so, by Euclid VI.17, AH:HD = HD:HT. Accordingly, the rectangles AH,HD and HD,HT are similar, and, by Euclid VI.20, AH,HD:HD,HT = $HD^2:HT^2$. But AH,HD: HD:HT = AH:HT, so AH:HT = $HD^2:HT^2$, or, conversely, HT:AH = $HT^2:HD^2$.

[12]It has been established above that $(AI^2 - AH^2):AH^2$ = HT,TN:QH^2 and that HT,TN = three-fourths HT^2. It has also been established that HQ^2 is one-fourth HD^2. Therefore, HT,TN:QH^2 = three-fourths HT^2:one-fourth HD^2 = $3HT^2:HD^2$, so it follows that $(AI^2 - AH^2):AH^2 = 3HT^2:HD^2$.

[13]See the end of note 10 above. According to Euclid, V.1, CA:AH = rectangle CA,AI:rectangle AI,AH. Accordingly, since CA:AH = $AI^2:AH^2$, then rectangle CA,IA: rectangle IA,AH = $AI^2:AH^2$. Now, according to Euclid, VI.20, polygons (including rectangles) are to one another in the duplicate ratio of their corresponding sides (i.e., as the squares on their corresponding sides). Hence, rectangles CA,AI and AI,AH are the relevant polygons, and AI and AH are the corresponding sides. If we then divide both sides by the common factor AI:AH, we get CA:AI = AI:AH, which leaves AI the mean proportional between CA and AH.

[14]It is clear from the figure that IA (which = AH + IH) + AH = 2AH + IH. If we divide both sides by three, we get one-third (IA + AH) = two-thirds AH + one-third IH. Since we just concluded that TH:IH = one-third (IA + AH):AH, it must also be as (two-thirds AH + one-third IH):AH.

[15]What Alhacen has accomplished to this point of the construction is to establish that M and L constitute endpoints of tangency for reflection-points G and B, respectively, on the mirror. In book 5, proposition 7 (Smith, *Alhacen on the Principles*, 404), it has been established that the ratio of the distance from the object-point to the center of curvature (designated as *a*) and the distance from the image-point to the center of curvature (designated as *b*) is the same as the ratio of the distance from the object-point to the endpoint of tangency (designated as *c*) and the distance from the endpoint of tangency to the image-point (designated as *d*), which translates to $a:b = c:d$. In the case of figure 6.4.3a, I is the object-point for both reflections, so IA (i.e., *a*) in both reflections is the distance from the object-point to the center of curvature. In the case of reflection from B, L is the endpoint of tangency, so IL (i.e., *c*) is the distance from the object-point to the endpoint of tangency. By construction, however, we know that IL:LH = IA:AH, so we have *c*: LH = *a*:AH, which reversed is *a*:AH = *c*:LH. H will thus be the image-point, and AH will be the distance from the image-point to the center of curvature (i.e., *b*), and LH will be the distance from the endpoint of tangency to the image-point (i.e.,

d). According to our initial proportion, then, $a:b = c:d$ = IA:AH = IL:LH. The same holds *mutatis mutandis* for M as the endpoint of tangency for reflection from point G on the mirror, so it follows from the proportion IM:MT = IA:AT, which is given by construction, that T is the image-point in that reflection (i.e., IA:AT = IM:MT = $a:b = c:d$).

The purpose of the long and rather involved line of reasoning preceding this paragraph is to locate endpoint of tangency M between I and H, which follows from the fact that IH:HT > IA:AT, which = IM:MT, by construction. It therefore follows that IH:HT > IM:MT, which means that IM < IH and MT is equal to or greater than HT. However, if MT = HT, then M must either coincide with H or lie at some point between A and T. If the latter, then IM > IH, which contravenes the initial condition that IH:HT > IM:MT. Therefore, MT > HT, and so it follows that M must lie between I and T. By the same token, IA:AT > IA:AH, so it follows that IM:MT > IA:AH, which = IL:LH, by construction. Therefore IM:MT > IL:LH, from which it follows that IL < IM and MT > LH. Hence, L must lie between I and M and thus also between I and H.

16The relevant theorem in this case is actually proposition 7 cited in the previous note.

17In other words, if IA:AH = AB:AP, then IA:AB = AH:AP. Likewise if IA:AT = AB:AR, then IA:AB = AT:AR, so AH:AP = AT:AR, from which it follows that AH: AT = AP:AR.

18That triangles OAY and GBS are similar follows from the fact that angle OAY = angle GAY + angle OAG, whereas angle GBS = angle XBS + angle GBX. But it has already been concluded that angle GAY = angle XBS and that angle OAG = angle GBX. Therefore, since they are composed of equal elements, angle OAY = angle GBS. Moreover, angles OYA and GSB are both right. Therefore, by Euclid, I. 32, the remaining angles BGS and AOY must be equal. By Euclid, VI.4, then, the two triangles are similar, and their corresponding sides are proportional.

19We have already established that $AH^2 + HB^2 = DA^2 + 2AH,HD + 2AH,DF$, so $AH^2 = DA^2 + 2AH,HD + 2AH,DF - HB^2$. By Euclid II.7, moreover, $AH^2 + HD^2 = 2AH,HD + AD^2$, or, conversely, $2AH,HD + AD^2 = AH^2 + HD^2$. Therefore, if we substitute the value for AH^2 derived earlier—i.e., $DA^2 + 2AH,HD + 2AH,DF - HB^2$—for AH^2 in this relationship, we get $2AH,HD + DA^2 = DA^2 + 2AH,HD + 2AH,DF - HB^2 + HD^2$. Adding HB^2 to both sides, we get $HB^2 + 2AH,HD + DA^2 = DA^2 + 2AH,HD + 2AH,DF + HD^2$. Dropping the common term $(AD^2 + 2AH,HD)$—which is equivalent to $(AB^2 + 2AH,HD)$—from both sides, we end up with $HB^2 = 2AH,DF + HD^2$.

20We established earlier that $AT^2 + TD^2 = AD^2 + 2AT,TD$, so $AT^2 = AD^2 + 2AT,TD - TD^2$; and we just established that $AT^2 + TG^2 = AD^2 + 2AT,TD + 2AT,DK$, so $AT^2 = AD^2 + 2AT,TD + 2AT,DK - TG^2$. Therefore, $AD^2 + 2AT,TD - TD^2 = AD^2 + 2AT,TD + 2AT,DK - TG^2$. Adding $TD^2 + TG^2$ to both sides, and subtracting common term $(AD^2 + 2AT,TD)$ from both sides, we end up with $TG^2 = TD^2 + 2AT,DK$.

21From above, we take AH as divided into 25 parts. We know from previous conclusions that HT < one of those parts and that HD > 5 HT, so AH:HD = 25:>5 HT. On the other hand, AT, which = AH – HT, is less than 25 and more than 24 of those parts. Finally, TD = HD – HT, i.e., > 5HT – HT, so TD > 4HT. Therefore AT:

TD [i.e., > 24: > 4HT] > AH:HD [i.e., 25: > 5HT]. Since, therefore, AT:TE′ = AT:TD, it follows that AT:TE′ > AH:HD.

[22]The reason for establishing that arc Q′D > arc GB will become clear in fairly short order. The proof that arc Q′D is in fact greater than arc GB, which occupies paragraphs 4.34-44, pp. 169-171, entails an enormous number of steps, but the logic behind it can be reduced to the following, in reverse order. Ultimately, what needs to be demonstrated is that Q′D:DA > BG:BA and, therefore, that Q′D > GB in view of the equality of DA and BA, which are radii of the mirror. This depends on proving that QH:AH = Q′D:DA and, moreover, that QH:AH > FK: OY. This, in turn, depends on proving that BG:GA = FK:OY, which, given the equality of GA and BA, is tantamount to BG:BA = FK:OY. Hence, if Q′D:DA > FK:OY, which is proportional to BG:BA, then Q′D:DA > BG:BA, from which it follows that Q′D > BG.

[23]In other words, since angle QMA is acute, QM must intersect circle FDB at some point to the left of M before reaching M itself. Hence, extension MZ of QM cannot intersect the circle to the right of point M, which is a roundabout way of saying that MZ can be a line of reflection. In essence, what we have done to this point is rotate mirror DGB in figure 6.4.3b to the left according to angle IAN while leaving line AGR fixed so as to carry line ABC the same distance in the same direction to coincide with line AMU. Points I, H, D, C, and B will thus sweep out arcs IN, HQ, FD, CU and BM. Hence, arc DF = arc BM, arc IN = arc CU, and line AFQN will be cut so that NA = IA, QA = HA, and FA = DA. Under those conditions, the form of point N will reflect along MZ according to the equality of angles NMU and UMZ, which are the respective angles of incidence and reflection. For any center of sight located on MZ, then, the image of N will appear at point Q, where the extension of line of reflection MZ intersects cathetus NA dropped from object-point N to the mirror's center of curvature at point A. It should be noted, however, that the construction and proof will apply whether M is located to the left or to the right of B, just as long as the resulting arc MB is equal to arc FD.

[24]It is at this point that the restriction on the length of ZD posed in paragraph 4.12 at the beginning of the proposition must be lifted. To understand this point and its implications, let us summarize the method of construction based on ZD, as represented in figure 6.4.3a, p. 100. Add it to radius AD of the mirror, bisect it at point H, and through that point form a circle centered on A. Bisect HD, form chord QH equal to it, and extend line AQ through Q. Find HT such that AH:HD = HD:HT, and connect QT. Extend line AQ well beyond Q, and find point I on ZD such that line IS dropped from it parallel to chord QH and touching the extension of AQ is equal to line QT. IS will thus be the chord of a circle with radius AI. Locate point M on line IT according to the proportion AI:AT = IM:MT, and locate point L on line IH according to the proportion AI:AH = IL:LH. Draw tangents MG and LB to the mirror from endpoints of tangency M and L. Hence, G will be the point of reflection according to which IG forms the line of incidence and TG the line of reflection along which the form of I is seen at T. B, on the other hand, will be the point of reflection according to which IB forms the line of incidence and HB the line of reflection along which the form of I is seen at Q. Then, in figure 6.4.3b, p. 101, find point M to the left of G such that arc BM = arc FD (= Q′D in figure

6.4.3a, with N replacing S in that same figure). According to symmetry, it follows that, just as IB and HB form the respective lines of incidence and reflection for point B of reflection, lines NM and QM will form the respective lines of incidence and reflection for point M of reflection. So point I will appear at point T for any center of sight posed to the right of G on line of reflection TG, whereas point N will appear at point Q for any center of sight posed to the right of M on line of reflection QM.

Let us extend ZD (= Z′D in figure 6.4.3b). As it increases in length, its half, HD, will increase commensurately, and likewise HD′s half, HQ, will increase along with FD, since both HQ and FD subtend the same angle. Moreover, as HD increases, so does HT, which is tied to it according to the proportionality AH:HD = HD: HT. Along with these increases, AI also increases, and with it NI, which subtends the same angle as HQ, FD, and BM (which equals FD by construction). Since the lengths of lines AI, AH, and AT determine the location of endpoints of tangency M and L and thus the location and size of angle GAB, any change in those lines will cause a change in that angle. The only constant throughout these changes is the equality of NI (= SI in figure 6.4.3a) and QT.

Even a moderate increase in the length of Z′D in figure 6.4.3b (= ZD in figure 6.4.3a), will cause a relatively significant increase in the size of angles BAG and GAM (and angle BAM composed of them) because of a significant increase in the length of lines AI, AH, and HT, as well as in the sizes of NI, QH, and FD subtending the same angle. However, as we increase HD and all its accompanying parameters, angle BAG increases faster than angle GAM so that a point is eventually reached at which angle GAB = angle GAM, as represented in figure 6.4.3c, p. 102. In order for that to happen, though, HD has had to increase dramatically enough to be almost equal to AH, which means that radius AD of the mirror has dwindled to virtual insignificance in relation to HD. Meantime, arcs NI, QH, FD, MB, and GB have increased in size, as have lines AQ, AN, AH, AI, NI, QH, HT, and QT. In fact, HT has become nearly as long as HD. But over the course of these changes the increase in the size of angles NAH and MAB has decelerated drastically because, as HD approaches equality with AH, chord QH, which equals half HD, approaches equality with half AH. Accordingly, the closer HD approaches AH in length, the more angle NQAHI, approaches constancy and the more NQ, AQ, IH, AH, and QT approach equality. One other crucial point emerges from this analysis: as angle BAM increases in size, line ABC moves clockwise toward the horizontal, which is to say that angle DAB approaches 90°. As that happens, lines AG and AM also move clockwise such that line AM moves somewhat faster than line AG, until finally the two reach a point at which angles BAG and GAM are equal. It therefore follows that, when ZD, and thus HD, is increased beyond the point at which it yields equality between angles GAB and GAM, angle GAB will begin to overtake angle GAM in size, but by tiny increments according to the ever-diminishing increase in the overall size of angle GAM.

[25]This is a special case in which ZD in the original construction is just the right length that arc NU = arc UZ, and arc IR = arc RZ. Thus, incident rays NM and IG are equal, respectively, to reflected rays MZ and GZ. Accordingly, as seen from center of sight Z, where the two reflected rays converge, image TQ will be the same length as its object-line NI, as demonstrated earlier.

[26]The explanation that line KG intersects line MZ because points K and M are "lower" than points Z and G, respectively, indicates the analysis was conceived with the original diagram (whether actually drawn or merely imagined) rotated 90° counterclockwise. In order to get BAG even minimally larger than GAM according to the construction outlined in note 25, it is necessary to lengthen ZD by so much that, within the scale of a page, the mirror dwindles almost to a point in relation to AH and QH. As represented in figure 6.4.3d, line of reflection QMZ barely grazes the mirror as it continues on to intersect the outer circle containing object-line NI. Accordingly, angle IAGR < angle ZAGR by a small amount. Meantime, reflected ray TGK continues to point K on the outer circle such that angle TAG = angle KAG. Thus, lines QMZ and TGK will intersect to the right of points M and G—at point L in this case. If a center of sight is placed at L, then, the entire image QT of object-line NI will be seen, and the two lines are equal by construction.

[27]As in the previous case, so in this one, the reflected ray TG for incident ray IG, when it is extended, intersects the outer circle at some point—O in this case—other than Z. Likewise, as angle BAG approaches equality with angle GAM, point O approaches point Z, and as it does, the intersection-point of reflected rays OG and ZM also approaches point Z.

[28]In other words, if the differential between angles BAG and GAM is small enough, the intersection-point of the two rays will lie somewhere to the right of reflection-points M and G, so this point will serve as a center of sight from which image TQ = object IN. As that differential increases, however, the intersection-point migrates toward reflection-points M and G and eventually beyond them to the left, as in fact is illustrated in figure 6.4.3e, where X marks that intersection. In that case, of course, the intersection-point can no longer serve as a center of sight.

[29]According to the conditions previously laid out, BAG < GAM. But BAG can be as small as we please and still fulfill that condition. Let BAG be as small as possible—i.e., virtually 0°. Hence, BAM = GAM, and since BAM = IAN, by construction, it follows that GAM = IAN. Consequently, if angle BAG > 0°, GAM < IAN. But NAZ = IAO + GAM. Hence, if GAM < IAN, NAZ exceeds IAO by less than IAN.

[30]According to Euclid, XI.23, two conditions must be met for a solid angle to be formed from the three angles IAO, NAZ, and IAN. First, any two of the angles taken together must be greater than the remaining angle. This condition is satisfied, because we just established that OAI + ZAN > NAI, OAI + IAN > ZAN, and ZAN + IAN > IAO. The second condition is that the three angles taken together sum up to less than four right angles. That the three angles under consideration fulfill this condition is clear from the following. If a tangent is dropped to a circle from any point on the extension of a diameter, the angle formed by the extended diameter and the radius intersecting the point of tangency will be less than a right angle. LAB in figure 6.4.3a, p. 100, is such an angle, by construction, so it is less than a right angle. But LAB = IAB, so IAB < a right angle. By the same token, MAG in figure 6.4.3a < a right angle, by construction. But MAG = IAG, so IAG is less than a right angle. Since NAZ in figure 6.4.3e = 2IAB, NAZ < two right angles, and since IAO = 2IAG, IAO < two right angles. But IAG = IAM + GAM, and, as we just determined, GAM = IAN when BAG = 0°. It therefore follows in this case that IAG = IAB. Hence, at its very largest IAO = NAZ + IAN. But IAO < two right angles, so NAZ + IAN < two right angles, so IAO + NAZ + IAN < 4 right angles.

[31]The solid angle resulting from this construction will therefore form a pyramid upon base NAI of the circle. Within that pyramid (as represented in figure 6.4.3f) edges AS, AI, and AN are equal, edge IS is equal to the line IO in the base-circle, and edge NS is equal to line NZ in the base-circle.

[32]The construction described to this point can be easily visualized according to figure 6.4.3g, p. 106. Imagine that all the lines in the figure are rigid filaments. Imagine further that arc IURZO lies flat on the ground and that the entire structure IAOGT in black is a flap lying on it and hinged along IA. The entire flap can thus be rotated toward S, and as it is rotated it will carry filament AG along with it. Then imagine that structure NAZMQ represents another flap atop IAOGT and hinged on NA. It too can be rotated toward S and, in the process, will carry filament AM along with it. Start by lifting this latter flap, then lift flap IAOGT, and bring them together until their respective sides OA and ZA come together. When they do, the two flaps will form a pyramid with its vertex at S, and all the angular relationships that obtained within them when they lay flat upon the base will also obtain when they assume that position.

[33]The argument to this point can be readily understood by recourse to figure 6.4.3h, p. 107. It has already been established that triangle SAT = triangle OAT and, therefore, that ST = TO, SA = AO, and AT is common. It has also been established that angle IAG = angle GAO, so AG bisects OT. G, however, is one of the two points at which OT intersects the sphere of the mirror on great circle DG. Hence, Y will be the corresponding point on TS where that line intersects the sphere of the mirror on great circle DYO′. The same analysis holds *mutatis mutandis* for the reflection from point M, so that triangle NAZ = triangle NAS in figure 6.4.3f, p. 105, and from there it follows that line AZ′Z″ bisects angle NAS, line QZ′ in triangle QAZ′ corresponds to line QM in triangle QAM, line SZ′ in triangle SAZ′ corresponds to line MZ in triangle MAZ, and therefore point Z′ on great circle O′Z′F corresponds to point M on great circle FDMB.

[34]Let NI and QH in figure 6.4.3k, p. 108, be the chords subtending arcs NI and QH, and let the sought-after perpendicular meet QN either at point P, between Q and N, or at point P′, beyond N. If P′I were the perpendicular, it would form triangle INP′, to which INQ would be an exterior angle. According to Euclid, I.32, the exterior angle of any triangle equals the sum of the opposite interior angles, which in this case would be IP′N and NIP′. Hence, INQ = IP′N + NIP′. But IP′N is supposedly a right angle, so IP′N + NIP′ > a right angle, from which it would follow that angle INQ, which is acute by construction, is also greater than a right angle. Hence, it follows that point P must lie below point N.

[35]Book 5, proposition 17, in Smith, *Alhacen and the Principles*, 414-415. According to the enunciation of that theorem, for any two object-points facing a convex spherical mirror, "the image-location of the point nearer the [mirror's] center will lie farther from the center of the [mirror] than the image-location of the point farther from the [mirror's] center." Since point N in figure 6.4.3k lies farther from centerpoint A of the mirror than does point P, image-point Q of point N must lie nearer A than image-point Q′ for point P, which is to say that Q′ must lie above Q, between it and P.

[36]While the conclusion of this paragraph is correct, the details of analysis are not. On the one hand, it is true that, if Z in figure 6.4.3m is the center of sight, M the point of reflection, and Q the image, line ZQ will intersect the circle at point M as well as at another point E to the left of M. It is also true that if a tangent is drawn from Z to the circle, it will touch it at some point V on arc ME. On the other hand, it is not true that that this tangent defines the endpoint of tangency. That point lies on the line tangent to the mirror at the point of reflection, XMY in the diagram, which extends below ZM. It is also true that no object-point at or below endpoint of tangency X can yield an image for Z because it cannot reflect to that point.

[37]Later in the theorem it becomes clear that Z must lie the same distance as E from center of curvature G.

[38]As with point Z, so with point H, it must be located precisely the same distance as E from center of curvature G. Alhacen's failure to make this specification until the next paragraph is somewhat puzzling, given the exquisitely systematic approach he normally takes to proof.

[39]As we established earlier, AG:GD = (AG:HM)(HM:GD). But we just established that AG:HM = AZ:ZH and that HM:GD = HE:ED, so, if we substitute the equivalent ratios, it follows that AG:GD = (AZ:ZH)(HE:ED).

[40]That this claim is patently false follows from reversing the construction and proof. It has been established in that proof that, when base line AGBD is cut according to the ratio AB:BD = AG:GD, any line drawn from A that passes above it through BE, DE, and GE will be cut according to that ratio. AZHT is one such line, but there is an infinitude of lines between it and AGBD that fulfill the stated condition. Thus, if we take AZHT as our base line and extend lines ET, EH, and EZ below it, any line (including AGBD) below AZHT that passes from A through those extensions will be cut according to the mandated ratio. The same will therefore hold for any line below AGBD that passes from A through the extensions of BE, DE, and GE.

Whether, in fact, this erroneous claim originates with Alhacen is unclear, because the text is problematic insofar as five of the manuscripts have *in quam* ("in which") instead of *numquam* ("never") in line 13. To choose *in quam* over *numquam* at this point, however, would have required such a tortured reading of the sentence that I felt compelled to plump for *numquam*, despite my conviction that Alhacen was far too good a mathematician to have made such a mistake.

[41]Thus, although the object AEB, the mirror, and the image QML are all concentric according to the construction, image QML will nonetheless be more sharply curved than the surface of the mirror, which in turn is more sharply curved than object AEB. It follows, therefore, that image QML does not take on the curvature of that surface. This point is of course obvious to anyone who looks at himself in a convex spherical mirror and notices that the outlying edges of the image appear to lie farther away than they should with respect to the center of the image.

[42]Alhacen's point here seems to be that the largest angle the perpendicular dropped to the plane from D forms with that plane is smaller than the largest angle formed by all other lines dropped from D to that plane, since those lines

necessarily form an obtuse angle on one side. To illustrate this point, let YZ in figure 6.4.8a, p. 111, represent the sphere from which the mirror is composed, G the center of curvature, and Y′Z′ a great circle on the surface of that sphere within the plane of arc AB. Let GD′ be perpendicular to that plane at point G. Let center of sight D be displaced to the side of D′, and let DX be the perpendicular dropped from D to the plane of arc AB. Accordingly, within the plane formed by DX and DG, which forms common section XGR with the plane of arc AB, angle DGR will be obtuse and thus greater than right angle D′G. No matter where D is displaced with respect to D′G, therefore, line DG will always form an obtuse angle on the plane of arc AB. It is also the case that, since D is displaced to the right of D′, it will be closest to point A and farthest from point B. Hence, AD < ED < BD. Moreover, with respect to the plane of arc AB, the obtuse angle DA forms with that plane will be smaller than the obtuse angle DE forms with that plane, which will be smaller yet than the obtuse angle DB forms with that plane.

[43]As will be evident from the analysis that follows, DG in this case must lie to the right of the perpendicular dropped to G on the plane of arc AB, as just illustrated in figure 6.4.8a.

[44]According to construction, BG = CG, and according to proposition 4, lemma 1, since B lies farther from D than C, endpoint of tangency L for point B lies farther from center of curvature G than endpoint of tangency M for point C. Hence, GL > GM, from which it follows that remainder CM > remainder BL.

[45]That QT, LM, and BC will intersect at point O when extended follows from the fact that the images of B and C are seen by a single center of sight, D. Hence, by book 5, proposition 7, GC:CM = GQ:QM and BG:BL = GT:TL. But those proportions are proportional, since they are based on the same center of sight, so it follows that GC:CM = GQ:QM = BG:BL = GT:TL. Thus, line GC is divided at Q and M in proportion as GL is divided at T and L, so, by proposition 6, lemma 3, lines QT, LM, and BC, when extended, will intersect at a single point.

[46]In other words, if center of sight D is displaced above perpendicular D′G so that the perpendicular dropped from it to the plane of arc AB intersects that arc at its midpoint, the images of the endpoints will lie precisely the same distance from center of curvature G, so the curvature of the resulting image will have the same orientation as that of the mirror, although it will be sharper, as will be shown in the following proposition.

[47]Although there is no indication at this point in the construction that object-line AB and normal DG, whose endpoint D represents the center of sight, do not lie in the same plane ABGD, it becomes clear later in the proposition that in fact they are to be imagined not to lie in the same plane. Accordingly, in figure 6.4.12, the plane formed by AG and BG—i.e., AGB—cuts the mirror along great circle SEZ (reading clockwise from S), and AB extended is tangent to the mirror at point E on that great circle. On the other hand, plane AGD formed by AG and DG cuts the mirror along a great circle containing arc PZ, whereas plane BGD formed by BG and DG cuts the mirror along a great circle containing arc PH. These two planes, AGD and BGD, therefore intersect along common section DPG, and they intersect plane AGB containing object-line AB along arc ZH to form triangle PZH on the

mirror's sphere. Although the subsequent analysis is based on this three-plane arrangement, that analysis will also obtain if AG, BG, and DG all lie in the same plane. This case is represented in figure 6.4.12a, p. 115, where the construction and analysis for this proposition is somewhat easier to follow than in figure 6.4.12.

[48]In book 1 of the *Sphaerica,* Menelaus of Alexandria (fl. c. 100 AD) shows the parallels between spherical and plane triangles. This work was well known to medieval Arabic mathematicians and was thus doubtless known to Alhacen. Suffice to say, when DG, AG, and BG lie in the same plane, as represented in figure 6.4.12a, p. 115, triangle HZP is conflated into the single arc PZH, with PZ < PH and HZ taken together.

[49]According to construction, of course, angle AGD = angle AGK, and DG = KG. Thus, the reflection of point A's form to K from point of reflection R′ will be perfectly equivalent to the reflection of point A's form to D from point of reflection R.

[50]The fundamental purpose of this rather elaborate construction is to establish M as the endpoint of tangency for B and N as the endpoint of tangency for A vis-à-vis center of sight D. From previous reasoning it follows that the extension of line NM will meet the extension of line IO connecting the images of A and B at a single point Q on AE. The image U of point Q, as seen from D, will necessarily lie on cathetus QG, and that cathetus lies below the two catheti AG and BG for points A and B. Hence, the image of Q will lie below those two images and thus below line IOQ connecting them. Since image IOU must be continuous, it follows that it must be curved in order to include all three points.

[51]Having arc HF′ = arc HE is tantamount to sliding line AZG in figure 6.4.12, p. 114, toward B along AE while, at the same time, pulling AE inward until it coincides with line BF. By the construction in paragraph 4.115 of proposition 12, ZP = ZY, but in the construction for figure 6.4.13, arc ZP (which = arc HP) = arc HS (which = arc ZY). So HS and HY coincide, which means that GL and GK coincide. Point Q is therefore the point from which the form of coincident points A and B reflects to coincident points L and K. Hence, points F and R′ in figure 6.4.12, which are the respective points from which the forms of B and A reflect to points L and K, coincide at point Q. By symmetry, then, points F′ and R in figure 6.4.12, which are the respective points from which the forms of B and A reflect to point D will coincide. Points A and B will therefore coincide, as will points C, M, and N and E, Q, and T. Accordingly, images I and O of A and B, as seen from D, will coincide at point O on cathetus AG (= BG). Let A be the endpoint on that cathetus. If we choose random points X and Y between A and Q in figure 6.4.13a, p. 117, and if we determine their respective images X′ and Y′, the composite image OX′Y′Q of line AQ will be curved, as predicted by the analysis. Substitute B for any of these points, and it follows that the image of AB will be appropriately curved.

[52]Thus, as represented in figure 6.4.13b, p. 118, HE > HF′, and ZP = ZY, according to the construction for figure 6.4.12. As before, M is the endpoint of tangency for B, whose form reflects to D from point F′ and to L from Q (which is point F in figure 6.4.12), so HF′ = HQ. Point R is the point from which the form of A reflects to D, and R′ the point from which the form of A reflects to K, so the two

reflections are perfectly equivalent, from which it follows that ZR = ZR′. Point N on the extension of the tangent to R′ is therefore the endpoint of tangency for point A. Accordingly, the extension of NM will intersect line AE at Q′. Point I is the image of point A, and point O the image of point B. Hence, according to previous reasoning, the extension of IO will intersect line AE at point Q′. When image-point U of point Q′ is determined for center of sight D, the composite image IOU of AB will therefore be appropriately curved.

[53]In figure 6.4.13c, p. 119, HE < HF′. As before ZP = ZY, HF′ = HQ, ZR = ZR′, M is the endpoint of tangency for B, and N is the endpoint of tangency for A. Thus, when extended, NM and IO intersect at Q′. When image U of Q′ is located, the resulting composite image IOU of A, B, and Q will be appropriately curved. The limiting case for this situation occurs when arc HE dwindles to 0° so that A and B lie on the same cathetus AG. In that case, point Q will lie where that cathetus intersects the mirror, so its image U coincides with it. Thus, the entire foreshortened image IOU of ABG will lie directly in line with it on cathetus AG, as was empirically determined in 5, 2.6 (Smith, *Alhacen on the Principles*, 387). IOU will therefore not be curved, as Alhacen points out at the beginning of the very next proposition.

[54]Actually, this stipulation was never made explicitly, although as we saw, the construction in proposition 12 implies that the visible line and the center of sight do not lie in the same plane.

[55]What Alhacen means by "slanting away from the eye" (*declinatio ex parte visus*), is open to interpretation, and the manuscripts are no help, since they have no diagrams to accompany this proposition. According to my interpretation, such a slant is defined by the angle formed on the side of the eye by the visible line and the line bisecting the angle formed by the two normals dropped from the center of sight and the object-point. Thus, in figure 6.4.14, the slant of object-line AB is determined by angle GCA. Since it is obtuse, AB is assumed to slant away from the eye.

[56]In order for the entire line AB to be invisible in the mirror, it would of course have to intersect normal DG so as to block all reflection to D. As Alhacen lays out the analysis, however, it seems necessary to leave an opening between this normal and line AB so that a supposed point of reflection R′ can be chosen on the arc between DG and CG such that it will yield an open line of reflection to D—the proof resting ultimately on the fact that this cannot be a legitimate point of reflection if R is a legitimate point of reflection. Despite the rather infelicitous articulation of this proof, Alhacen's basic point seems to be as clear as it is banal: if an object gets in the way of its reflection, it will not be seen in a mirror.

[57]What Alhacen seems to have in mind here is that, if it were not blocked by the remainder of line AB, the form of A would pass through X after reflection from B, whereas if it were able to reflect from point R′, the form of A would also pass through X on its way to D. Therefore, two distinct forms of A would reach D, which is impossible.

[58]The points made in these two paragraphs are illustrated in figure 6.4.14b, p. 120. First, if line AB from the previous figure is shortened to AC, then its form

can be reflected to D from B, but only endpoint C of that line will be seen in the mirror. Since the line is physical, that endpoint will have some dimension, albeit tiny, so its image will have some dimension, however minuscule. Moreover, if that slanted line is transferred to position A′C′ between normals FBG and DG, it will be visible in the mirror from D, and the curvature of its image will depend on the angle it makes with FBG—i.e., the closer it approaches FBG, and thus the closer it approaches a perpendicular position with respect to the mirror, the less curved it will appear, which is the point of paragraph 4.127.

[59]What Alhacen is getting at here is not clear because it is difficult to know precisely what is meant by *conterminabilis*. I have chosen to translate it somewhat vaguely as "bordering on," although it could mean "neighboring" or even "sharing the same endpoints." Accordingly, my interpretation of the passage is as follows. Let DB in figure 6.4.14c, p. 121, represent the line of reflection from the previous examples, and let arc EBC be the visible portion of the mirror defined by tangents DE and DCK dropped to the mirror from D. FH is the one line parallel to DB that touches the mirror on the right side in the invisible portion. Anything that borders on line FH below it (including FH itself) will therefore be invisible. Thus, ML is obviously invisible. On the other hand, although not everything that borders on FH above it will be visible, everything that does so and extends above extension CK of the tangent dropped from D will be visible. Accordingly, the segment of M′L above CK will be visible in the mirror, the point where M′L and CK intersect being the endpoint of tangency.

[60]My best guess at the meaning of this passage is illustrated in figure 6.4.14d, p. 121, where the two parallels are taken to be the line of reflection itself and the line parallel to it that is tangent to the circle of the mirror at point F. Accordingly, if the object-line is slanted toward the center of sight, as represented by XY, and if it does not touch either of the parallels, it will be visible in the mirror from center of sight D. On the other hand, if it slants away from the eye, as represented by line X′Y′, it will be visible as a line, rather than merely according to its endpoint, so long as it does not coincide with the line of incidence extending from its endpoint X′ to point B of reflection. If that is the case, then it is unclear what Alhacen means by stipulating that it will be visible as long as the line dropped from its endpoint parallel to the parallel lies above that parallel.

[61]That a line beyond the center of sight cannot be seen, except perhaps for terminal portions, is presumably due to the inability of its form, or at least the main central part of its form, to reach the mirror because it is blocked by the head. By the same token, any line intersecting the visual axis and thus lying between the center of sight and the mirror will block its own form from reflecting back to the center of sight. However, a line parallel to the visual axis will be visible, as long as it extends above the visible portion of the mirror.

[62]The more completely the visible line is exposed to the mirror vis-à-vis the center of sight, the longer its image will appear and, therefore, the more "clearly" it will appear. This exposure varies with the slant, so the greater the slant, to put it in Alhacen's terms, the more complete the exposure. Wherever the object-line is placed with respect to the mirror's surface and the center of sight, its greatest

exposure occurs when the line from its midpoint to the center of curvature is perpendicular to the mirror's surface. As to the curvature of the resulting image (whether of a straight line or of an arc), in saying that it accords with the curvature of the mirror Alhacen does not mean that it takes on the actual curvature of the mirror. This he has in fact denied in proposition 8 above. What he means, instead, is that the sharper the mirror's curvature, the sharper the curvature of the object seen in it. Thus, for a line of a given size, the smaller the sphere from which the mirror is composed, the more sharply curved the object-line will appear in it.

[63]In other words, when the visual axis of either eye or the visual axes of both eyes lie in the same plane as the object-line, the object-line will appear only lengthwise in the mirror, and it will be commensurately foreshortened. As the center or centers of sight move to face the object-line more directly, however, the image is seen under a larger visual angle, so it appears less foreshortened and therefore clearer.

[64]It is worth noting that when object-line HE is parallel to line DG, as represented in figure 6.4.15, p. 122, or when it slants above that position, lines HC, TK and ZL will intersect to the left of cathetus HG, and this will continue to be the case until object-line HE achieves the slant in figure 6.4.15a, p. 122, at which point HC, TK, and ZL are parallel. Thus, when HE inclines below its position in figure 6.4.15a, as represented in figure 6.4.15b, HC, TK, and ZL will intersect to the right of cathetus HG.

It is also worth noting that Alhacen does not address the case in which the extension of object-line HE intersects the mirror's surface at some point to the right of B. In fact, this case has already been addressed implicitly in propositions 12 and 13, where the proofs hold whether the normals from the center of sight and the endpoints of the object-line lie in different planes or in the same plane (see note 47 above). Hence, the primary difference between this proposition and propositions 12 and 13 is that in this case object-line HE (i.e., AB in propositions 12 and 13) does not touch the circle of the mirror at any point. Otherwise, the proof is essentially the same, point C in proposition 15 serving the same function as points Q and Q′ in propositions 12 and 13, respectively.

[65]These errors include all the ones stemming from the fact that reflection weakens light and color, both of them being further affected by the mirror's color. It also includes image-reversal. See paragraphs 3.5-3.12, pp. 163-164, for a full account of these errors.

[66]If the Latin is to be taken literally, line EU should fall below (*sub* in the text) both axis DH and point D on line DB. As represented in figure 6.5.16, however, although line EU does fall below point D on line DB, it falls in front of rather than below the axis. Presumably, then, the diagram Alhacen had in mind was rotated 90° clockwise so that EU would actually lie below the axis. Consequently, in this and succeeding theorems I have rendered *sub* as "outside" rather than "below."

[67]Whether DL intersects the plane of circle ESP inside the circle itself or outside it depends upon the slant of the elliptical section; the more acute CDL becomes, the farther point F migrates toward point S on the circle until, finally, it falls outside that circle. DL is, of course, the major axis of the ellipse, and BO is its minor axis.

[68]According to Euclid, VI.8, TM is the mean proportional between DM and MC. Therefore, DM,MC = TM^2. However, it has just been established that TM = FE, so by substitution it follows that DM,MC = TM,FE.

[69]In other words, if angle LED were right, then according to Euclid VI.8, rectangle DF,FL should be equal to EF^2. But we have just established that DF,FL > EF^2, which means that EF^2 is equal to some rectangle consisting of FL and some length D′F < DF, so D′ will fall somewhere between D and F. Hence EF^2 = D′F,FL, which fulfills the condition for Euclid VI.8, which in turn means that angle D′EL is right. If so, then angle DEL must be obtuse, since it is subtended by a longer side (i.e., DL) than angle D′EL. Hence, EU, which forms a right angle with EL, by construction, must pass through point D′ and thus outside point D.

[70]The point of this proposition is actually quite simple and can be easily understood in the context of figure 6.5.16a, p. 125, where the problem is viewed within the plane of the elliptical section. Let BD be the minor axis of that section and DFL the corresponding major axis. With N as the object-point, assume that its form reflects from point B at the end of the minor axis and therefore within the plane of circle BO passing through that point. Let V be the center of sight, so that VB is the reflected ray. The problem is to determine image-location I, which will lie at the intersection of the cathetus dropped from point N to the ellipse and the extension of line of reflection VB. The point of this theorem is to demonstrate that the cathetus, NEU, which is perpendicular to tangent QEL at point E, will intersect the normal BD dropped to point of reflection B at some point U beyond point D such that the cathetus will lie below line BD. Furthermore, it should be clear that cathetus NEU bypasses the axis of the cylinder in front of it, since point D′ lies on segment DL of the major axis, which diverges away from axis DH in figure 6.5.16. On the other hand, if N were the center of sight and V the object-point, then the relevant tangent would be Q′E′L′, and the cathetus VE′U would fall above line BD and would bypass the cylinder's axis behind it.

[71]That is, C lies at the intersection of planes QLHK and EGBA and therefore lies in both planes. QL is, of course, the cathetus, so image-point C lies where it is intersected by the extension of line of reflection EB.

[72]In other words, since points T and H are equidistant from point Q, their reflection within their respective planes of reflection will be perfectly symmetrical, which means that angle of incidence TGN in T's plane of reflection will be equal to angle of incidence HAZ′ in H's plane of reflection, and so will respective angles of reflection EGN and EAZ′. This in turn means that image-locations I and S will be equivalently situated within their respective planes of reflection, so that TI = HS, which means that straight line IS will be parallel to line TH, as well as to axis ZK and line of longitude AG. It bears noting, moreover, that the two catheti, TIX and HSP will both lie behind axis ZK insofar as the ellipse formed on the cylinder by plane of reflection TGE will tilt downward toward E, whereas the ellipse formed on the cylinder by plane of reflection HAE will tilt upward toward G, both ellipses being tilted at identical angles. It therefore follows from our analysis in note 70 above that both catheti will bypass axis ZK behind it.

[73]That the Latin text seems to be defective at this point was noted by Risner, who adjusted it to read *quia perpendicularis ducta a puncto C ad punctum sectionis*

lineae IS et superficiei circuli est valde parva ("because the perpendicular dropped from point C to the intersection-point of line IS and the plane of the circle [BF] is extremely small"). Unlike the reading in the Latin text, this adjusted one makes clear and obvious sense.

[74]This claim could be taken in two ways. On the one hand, it might be understood to mean that, if TH is slanted with respect to AGB within plane TOH, its image will become increasingly curved. On the other hand, it might be understood to mean that, if TH is slanted so as to lie in a different plane from ABG, its image will become increasingly curved. Both claims are true, but within the context of the proposition, the second alternative is more likely the one Alhacen had in mind. That image ICS of straight line TH is somewhat curved according to the conditions set in the proposition is empirically false, but Alhacen is forced to that conclusion by the cathetus-rule. According to his analysis, however, the image's curvature is virtually undetectable not only because it is so slight, but also because image ICS faces the eye directly so that it and the center of sight lie in the same plane.

[75]In other words, if the object-line is posed somewhat to the side of the normal dropped to the mirror from the center of sight, then a small portion of the nearer end of that line will be visible in the mirror through the opening between the object's endpoint and the aforementioned normal.

[76]In the hope of making this proposition easier to follow, I have provided three views of the construction. At the top is a three-quarter view from above, at the lower left is a side view from the left, and at the lower right is a bird's eye view.

[77]Angle MBL in triangle MBL = angle BED by construction, but angle BED = angle MEB in triangle MEB, and angle BMLE is common to both triangles. Therefore the third angles BLM and EBM, respectively, are equal, so, by Euclid, VI.4, the two triangles are similar, leaving their respective sides proportional. Accordingly, EM:BM = BM:ML. Since BM is the mean proportional between EM and BM, it follows from Euclid, VI.17 that EM,ML = BM^2.

[78]That angle BDM > angle ZDM can be shown as follows. From point M in the top diagram of figure 6.5.19, drop line M*x* perpendicular to CD, and from point *x* on CD erect perpendicular *xy* to intersect DB at point *y*. M*y* will therefore be the hypotenuse of right triangle M*xy* and will therefore be longer than M*x*. Since M*y* subtends angle BDM, while M*x* subtends angle ZDM, it follows that angle BDM > angle ZDM because it is subtended by a longer line. By the same token, since sides BD and DM of triangle BDM are equal to sides ZD and DM of triangle ZDM (BD and DZ being radii of the circle), it necessarily follows that side BM subtending larger angle BDM is longer than side MZ subtending smaller angle ZDM.

[79]It has just been established that MZD > MDZ (i.e., EDZ), and it was established at the end of 5.36 that MZL > ZED. Therefore, DZL, which = MZD + MZL, is greater than EDZ + ZED.

[80]In other words, if the form of point Q, which lies between F and B on line FNQB, were to reflect to E from some point Z′ on AZ above Z, the straight line connecting Q and that point would necessarily intersect line FZ, so the form of that intersection-point would reflect from both Z and Z′, which is impossible.

[81]In order to conserve space, I have not extended the two normals HU and TU to their actual intersection-point at U. Because of the confusion of lines, moreover, I have not attempted to represent normal TU in the top diagram of figure 6.5.19 and have thus not represented its intersection with reflected ray EG.

[82]Although all the manuscripts, as well as Risner, have *spericis* rather than *columpnaribus*, the comparison that follows is clearly between conical and cylindrical mirrors, not between conical and spherical mirrors. Moreover, in the introductory paragraph to chapter 9 on concave conical mirrors, the comparison is explicitly between those sorts of mirrors and concave cylindrical ones.

[83]As Alhacen explains in 4, 5.43-45 (Smith, *Alhacen on the Principles*, 341-342), if the plane of reflection cuts a conic section on the cone's surface, there can be one or at most two points of reflection on that section. There will be one only if the planar cut is perpendicular to the line of longitude, in which case the point of reflection lies where the axis, or major axis, of the conic section intersects the line of longitude. Otherwise, there will be two points of reflection, and each of them will lie at the intersection of a line of longitude and a line extended orthogonal to it from the section's focus, as determined by the intersection of the cone's axis and the section. In all cases, therefore, the appropriate reflection-point(s) will lie on lines passing through that focal point. As represented in figure 6.6.20, the section is an ellipse, point D is one of its foci, and point E is where line DE extended from that focal point to line of longitude AO is orthogonal to that line of longitude. If it is the only such point, then ED is the major axis of the ellipse. If it is one of two such points, then there will be a complementary point of reflection on arc FB of the ellipse lying the same distance as E from endpoint F of major axis FD.

[84] There is a general consensus among the manuscripts at this point in the text that the relevant proposition (*figura* in the Latin text) in book 5 is the nineteenth; there is no such consensus among them at the relevant spot in the text of book 5. Several, but not all, of the manuscripts give number-designations for the figures that accompany particular propositions. Among those that do (i.e., O, L3, E, and C1), O and L3 list it as 28 (see folios 65r and 74r, respectively), while E and C1 list it as 18 (folios 118r and 102r, respectively). This lack of agreement over the appropriate figure-designation reinforces the point made earlier in Smith, *Alhacen on the Principles*, cx, that such designations are totally unreliable as guides to the specific propositional structure of the *De aspectibus*.

[85]What Alhacen set out to do in this lemma was, of course, to show for convex conical mirrors what he showed for convex cylindrical mirrors in proposition 16, lemma 5: namely, that for a given point of reflection within a given plane of reflection, the normal, or cathetus, dropped from a given object-point will intersect the normal dropped through the point of reflection at some point outside the axis of the mirror. Thus, in figure 6.6.20, if H represents some object-point within the plane of conic section BFZ, and if HZ is normal to the section at point Z, then for some center of sight poised within the same plane on the other side of point E of reflection, H's image will lie at the point on ZX inside the conic section where the reflected ray meets cathetus HX.

[86]In other words, since RF is parallel to TZ, by construction, and since EF is parallel to OH, also by construction, and since OZ is a straight line within the same

plane as those two lines, then angle ZFR = alternate angle OZT, while angle FRZ = alternate angle TZR. But angles OZT and TZR are equal, by construction, so angles ZFR and ZRF are equal.

[87]To this point, proposition 22 is essentially a reprise of proposition 21 with some crucial differences that indicate a change of translators between the two propositions, a change that remains in effect for the remainder of book 6, as well as book 7. At the gross level, of course, there are certain instructions in this portion of proposition 22 that are missing in proposition 21, such as the formation of conic section BEG′ on the cone's surface (paragraph 6.25) and the extension of line AU through the point at which HO intersects the circle passing through Z (paragraph 6.22). There is also the change in lettering between the two propositions, translator 2 substituting R and K in proposition 22 for C and Q in proposition 21. Another noticeable difference between the two is that in proposition 22 and the succeeding text of book 6, angles are often referred to by vertex-designation only (i.e., *angulus H*), whereas in the entire text of book 6 up to proposition 21, angles are invariably spelled out completely (i.e., *angulus AHZ*). Beyond these gross variations, moreover, there is a host of stylistic differences that indicate a change of translators. For one thing, translator 2's sentence-structure tends to be choppier than that of translator 1. Accordingly, he often strings together long successions of clauses beginning with "et," "ergo," and "tunc." Unlike translator 1, as well, translator 2 makes frequent and consistent use of the hortatory subjunctive (i.e., *extrahamus lineam AB*—"let us extend line AB") rather than the jussive (i.e., *extrahatur linea AB*—"let line AB be extended" or "extend line AB"). Most telling, however, are the differences in phrasing and vocabulary between the two translations. Whereas translator 1 occasionally uses the phrase *quocumque modo* ("randomly" or "at random"), translator 2 uses it more often, adding *sit* or *fuerit* (*quocumque modo sit/fuerit*). Although translator 1 uses the phrase "est sicut" (or simply "sicut" with the "est" understood) to link two proportions in a ratio (i.e., *AG ad AN sicut GI ad IN*), he almost never uses it to mean *est equalis* ("is equal [to]"). Translator 2, on the other hand, uses *est equalis* and *est sicut* interchangeably throughout the remaining portion of book 6. As far as simple vocabulary shifts are concerned, a few salient examples should suffice. Whereas translator 1 always uses *exterioris* to mean "convex" (e.g., *in speculis spericis exterioribus*—"in convex spherical mirrors"), translator 2 always renders "convex" as *convexus*. Whereas translator 1 uses the term *error* for "misperception," translator 2 uses the terms *deceptio* and *fallacia*. Whereas translator 1 uses the form *columpnaris* for "cylindrical," translator 2 invariably uses *columpnalis*. Whereas translator 1 never uses *nam*, translator 2 uses it frequently. Whereas translator 1 never uses *tunc*, translator 2 often interchanges it with *ergo*. Whereas translator 1 renders "chapter" as *pars*, translator 2 renders it as *capitulum*. Whereas translator 1 almost invariably renders "to reflect" and "reflection" as *referre* and *reflexio*, translator 2 overwhelmingly prefers *convertere* and *conversio*. And finally, whereas translator 1 always renders "section" (as in "conic section") as *sectio*, translator 2 always renders it as *sector*. For further discussion of this issue, see the section on manuscripts and editing, pp. xlv-xlviii above.

[88]Because the construction that follows is so difficult to represent in three dimensions (as in the top diagram of figure 6.6.22a), I have provided a bird's eye view of the construction for the plane of reflection containing R, E, and N in the lower diagram, that view being from directly above the plane of the conic section.

[89]This follows from proposition 20, where it is demonstrated that the normal to point C will fall between CD and CZ′, both of which lie in the plane of the conic section. Hence, angle DCZ′ must be greater than a right angle.

[90]The Latin text I have translated as "and if line AON lies on some visible object" actually reads *et [si] linea longitudinis fuerit in aliquo visibili* in the majority of the manuscripts. Translated literally (i.e., "and if the line of longitude lies on some visible object"), this phrase makes no sense, so I have adjusted it accordingly. Risner went even further, rephrasing the Latin to read *et [si] forma alicuius visibilis reflectatur a linea longitudinis*, but this adjustment is more extreme than necessary. That P is in fact the image of O follows from the fact that lines ZH and UH are equipoised within circle ZU such that they both intersect that circle where it intersects the conic section formed on the cone by plane of reflection OZR, which contains those two lines. Thus, cathetus OUH is normal to that conic section, just as ZH is normal to it, by construction.

[91]Alhacen thus uses the same reasoning here that he used in the case of convex cylindrical mirrors to explain why the image, although curved, appears essentially straight: i.e., that its convexity faces the eye directly; see note 74 above.

[92]The notion of "compound misperceptions" (*fallacie composite*) crops up later in paragraph 7.108, p. 221. This presumably refers to the full complement of individual misperceptions that occur as a group in reflection. One group of such misperceptions is due to reflection itself, regardless of the shape of the reflecting surface—i.e., misperceptions arising from the intrinsic weakening of reflected light and color as well as from the mingling of the mirror's color with that of the object's form. The other group is specific to the shape of the reflecting surface and therefore involves not only distortions of shape, size, number, and distance, but also image-reversal or inversion. Note, by the way, that the term *fallacia* is used here for "misperception." This term is apt, because, as Alhacen explains it in book 2, chapter 3, the act of perception involves a low-grade syllogistic or deductive process. If the conclusions drawn from that process are false, then a fallacy occurs.

[93]The Latin text here reads *in omnibus speculis convexis et superficialibus*, which is open to interpretation because of the phrase *et superficialibus* ("and to superficial [mirrors]"). Given the context, I have chosen to intepret "superficial" as "plane."

[94]See paragraph 7.6 below for Alhacen's understanding of "the arrangement of parts" and its distortion in concave mirrors.

[95]See book 5, proposition 32, in Smith, *Alhacen on the Principles*, 446-449.

[96]The situation described here is a gross approximation of that in which an eye looks at itself in the mirror when the eye's surface lies between the center of curvature and the mirror's surface. Thus, MN can be taken to represent a cross-section of that surface within the plane of great circle BUG on the mirror.

[97]That the entire image falls outside arc BG follows from the fact that the image of any point on line MN will lie beyond the reflecting surface. Thus, as illustrated in figure 6.7.23, if some point X is chosen randomly on segment TN of that line, its form will reflect to T from point R along line of reflection RT. When that line of reflection is extended, it will meet cathetus AX well outside the mirror, and the closer to T the point that is chosen, the farther beyond the mirror's surface its image will lie.

[98]In other words, the planes containing triangles KGA and GBA will cut great circles on the surface of the sphere from which the mirror is composed.

[99]In other words, if LH and TK were two facing objects, with O a viewer poised between them, then from O's perspective when facing object KT, point K would be seen on the object's left-hand side, and point T would be seen on its right-hand side. By the same token, if the viewer turned to face LH, point L, which is the image of K, would be seen on the left-hand side of LH, and vice-versa for object-point T and its image H. Thus, object and image would correspond perfectly in their respective left-right orientations, and they would both be upright.

[100]I take the sense of this peculiar phrase *in linea in qua est de lineis radialibus* to be that, as before, if NU and MR were to represent facing objects, with center of sight O between them, then if O were to look directly at NU, rays OU and ON along which NU would be seen would correspond in left-right orientation to rays OM and OR along which MR would be seen.

[101]Line HTZ is in fact meant to represent a line on the surface of the eye, T presumably lying at the center of the pupil. Being straight, that line cannot actually be on the surface, so it is to be taken as a virtual rather than a real representation of the situation. As it turns out, point T's only function in the analysis is to anchor points H and Z at equal distances from line DE along a line perpendicular to DE. Suffice to say, this proposition is intimately related to proposition 23, where cross-section MTN in figure 6.7.23, p. 132 is equivalent to cross-section HTZ in the figure for this proposition—the main difference between the two situations being in the placement of that cross-section with respect to the center of curvature.

[102]Although somewhat vaguely put here, the point of this stricture is clear enough: A and H must be disposed in such a way that the angle formed by incident ray AH and reflected ray AE contains centerpoint G of the mirror so that the normal AG will bisect that angle. The crucial thing, in fact, is that the object-point and the center of sight flank the mirror's center of curvature—a point made earlier by Alhacen in book 5, paragraph 315 (in Smith, *Alhacen on the Principles*, 448-449).

[103]That GH > GK follows from the fact that AG bisects angle HAK in triangle HAK and cuts base HK at G. Thus, by Euclid VI.3, HA:AK = GH:GK. But HA > AK, so GH > GK. The same reasoning applies to triangle ZBL and the bisection of angle ZBL by GB, which cuts base ZL at G.

[104]The situation envisioned in this proposition is essentially the same as that in propositions 25-27, where the center of sight is posed behind the visible object (in this case the surface of the eye) so that the resulting image is formed between the center of sight and the reflecting surface and thus appears diminished and inverted.

[105]That FK > KA follows from the fact that triangle EFA is isosceles, because sides EF and EA are radii of the circle, so base angles EFA and EAF are equal. Therefore, since FK intersects EA between E and A, angle KFA < angle EAF. Consequently, in triangle KFA, side FK subtends a larger angle than side KA, so, by Euclid, I.19, FK > KA, and *a fortiore* it is longer than KE, because KA > KE by construction.

[106]In other words, we know that angles FEK, EFK, and FKE = two right angles. But angle FEK < angle EFK, so FKE + 2 EFK < two right angles. Since EFG = EFK, by construction, it follows that EFG + EFK = 2 EKF, so FKE + EFK + EFG < two right angles, from which it follows that EK and FG will intersect on the side of G.

[107]In this case, the citation of "figures 27 and 28" applies to actual figures that accompany the relevant section of book 5, chapter 2, which consists of proposition 34 in Smith, *Alhacen on the Principles*, 450-451. Here again, we encounter some discrepancy between the citation in the text at this point and the denomination of the figures in book 5. In O and L3, the figures are numbered 38 and 39 (folios 69r-v and 78r, respectively), whereas in E and C1 they are numbered appropriately as 27 and 28 (folios 106v-107r and 124r, respectively). Thus, the enumeration in E and C1 accords with the citation at this point in the text, although as we saw in note 84 above, this was not the case earlier. The point of book 5, proposition 34 is to demonstrate that, if the visible point and the center of sight lie on intersecting diameters inside a great circle on the mirror, there can be reflection from arc AD subtended by angle AED and from arc OB subtended by angle OEB on the opposite side of the circle in figure 6.7.29, but not from arcs DO or AB. Since in this case the mirror does not extend beyond O or B, according to specification, arc OB on the opposite side of arc AD is irrelevant, leaving arc AD as the only possible area of reflection in great circle BADO.

[108]The relevant theorems in this case are propositions 39 and 40 in Smith, *Alhacen on the Principles*, 459-60. As to numerical designations in the manuscripts, L3 fails to give any at this point, but O numbers them 47 and 48 (folio 71r). E and C1, on the other hand, label them in accordance with the citation here—i.e., as 35 and 36 (folios 127v and 109r, respectively). The ultimate reason that there can only be one reflection on arc AD in figure 6.7.29 is that object-point R and center of sight Z are poised on their respective diameters such that angle REZ facing arc AD is greater than two right angles, whereas if it were less than two right angles—as would be the case with respect to the missing portion of the mirror on arc BO—there could be as many as three points of reflection.

[109]The text is problematic at this point because "K" evidently designates two different points along the line extended to the mirror from G. In the initial part of the paragraph, it designates the point where that line intersects circle EGZ, whereas later it evidently designates the point where that line intersects the mirror's surface. That being so, the point of the argument here is clear. Since angles EKG and GKZ subtend equal arcs in circle EGZ, they are equal, so it follows that angles EK'G and GK'Z in circle ABD are not equal. Moreover, in the two triangles EK'K and ZK'K, angles EKK' and ZKK' are equal, whereas angle K'EK < angle K'ZK, so it follows that angle EK'K (i.e., EK'G) > angle KK'Z (i.e., GK'Z).

[110]The argument here can be easily understood by recourse to figure 6.7.31b, p. 139, which is abstracted from figure 6.7.31a. If we think of PH as a hinge, and

arc PK′H with constituent lines GNQ and EK′Q as a flap, then if we rotate that flap upward on PH until lines GNQ and EK′Q assume the respective positions GCS and EK″S, all the constituent points on the flap will remain in place. Thus, C corresponds perfectly to N, K″ to K′, and S to Q, so if the form of N reflects to E from point K′ on arc PK′H, the form of C will reflect to E from point K″ on arc PK″H. By the same token, if point Q is the image of N within plane of reflection EGQ, point S will be the image of C within plane of reflection EGS. And the same holds by symmetry for point R within plane of reflection EGO.

[111]As presented in the manuscripts, this proposition is marred not just by a welter of conflicting letter-designations, but also by a few complete mis-designations. The gist of the theorem is evident enough, however. In the construction at the beginning, Alhacen determines the image-locations for the three points of reflection A, B, and D in the arc on the mirror subtended by angle EGZ. The construction itself is based on the simplest possible case, in which center of sight E on diameter EG and object-point Z on diameter ZG are equidistant from center of curvature G and lie closer to the mirror's surface than they do to G. On that basis, as we have seen earlier, the points of reflection are easily determined by the intersection of circles EGZ and ABD. Once he has established the three image-locations F, M, and L for points of reflection A, B, and D, respectively, he adds the reflection for N on line MG from point K′ and then determines its image-location Q on that same line.

The next phase of the analysis is based on passing a plane perpendicular to circle ABG along line MG, selecting points C and R within that plane, and determining the image-locations of C and R with respect to Z and N within that plane. On that basis, Alhacen shows that, if the viewer confines his view to the portion of the mirror limited to the arc containing points D and K′ of reflection, the images SQO and SLO of lines CZR and ZNR, respectively, will appear concave to the center of sight at E.

In the final phase of the analysis, Alhacen determines the fourth point I from which the form of Z will reflect to E from the arc on the opposite side of ABD and locates the resulting image at T′. Then, if the viewer faces the entire portion of the mirror defined by arc ABDI, line CZR will have four images, all of them containing image-points S and O of points C and R, and each of them differentiated by whether the resulting image-line passes through point M, point L, point T′, or point F. Whichever the case, that line is concave with respect to the center of sight, hence the conclusion—which is limited to the analysis of line CZR—that straight lines have several concave images.

[112]Contrary to the apparent sense of the Latin wording here, angle EDG is meant to be several times smaller than angle ADE rather than the difference between ADG and ADE, which is, after all, EDG itself.

[113]This follows from the fact that angle ZDK = angle KDT + angle ZDT. But ZDK is exterior to triangle HDZ, so, by Euclid, I.32, it is equal to the sum of the opposite interior angles DZH and DHZ. But DZH = KDT, by construction, so remainder ZDT of external angle KDZ = remainder ZHD of the two opposite interior angles of triangle ZHD.

[114]LZH = BDK, by construction. But BDK = BDT + KDT, and LZH = LZD + DZH. Therefore, since DZH = KDT, by previous conclusions, LZD = BDT. Consequently,

LZD + BDZ = BDT + BDZ = TDZ, which is less than two right angles, so LZ and DB will intersect on the side of BZ.

115That angle LMD is right follows from the equality of triangles DML and DHL and therefore the equality of their respective angles. Thus, as we established earlier, angle LHD in triangle DHL is right, so its corresponding angle DML in triangle DML must be right as well. From that fact it follows that the circle must pass through M, since it forms a right angle on diameter DL.

116The rationale behind this conclusion is as follows. From Euclid, III.27, we know that angle FRM = angle FDM, since both angles lie on the circumference of circle FDZ and are subtended by the same arc FM. From Euclid,VI.33, moreover, we know that the angle with its vertex at the circle's center and subtended by a given arc on the circle is twice the angle with its vertex on the circle's circumference and subtended by the same arc. Therefore, the angle at the center of circle FDZ subtended by arc FM is twice angle FDM, whose extension to circle ABG is FE. But angle FDE is at the center of circle ABG. Therefore, in relation to circle FDZ, arc FM is double arc FE in relation to circle ABG—or, to put it another way, arc MF in circle FDZ occupies twice as much on the circumference of that circle as arc FE does on the circumference of circle ABG.

117The relevant theorem is book 5, proposition 33, in Smith, *Alhacen on the Principles*, 449-450, which is numbered 26 in C1 and E (folios 106v and 123v, respectively) and 36 in O and L3 (folios 69r and 77v, respectively). The logic underlying these proportionalities is as follows. Let U and O in figure 6.7.32a, p. 141, be object-points whose forms are reflected to center of sight H from points F and B, respectively, on the mirror. Q, which lies on the extension of line of reflection HB will be the image of O, and N, which lies on the extension of line of reflection HF will be the image of U. QD is therefore the cathetus dropped from both image-points. Draw tangent BT from B to intersect QD at T, and draw tangent FT′ from F to intersect QD at T′. According to Alhacen's designation in book 5, proposition 6 (Smith, *Alhacen on the Principles*, 403), T is the endpoint of tangency for reflection-point B, and T′ is the endpoint of tangency for reflection-point F. Accordingly, in proposition 33 of the fifth book, Alhacen demonstrates that DO:DQ = OT:TQ and, by extension, that ND:NU = T′N:T′U. Therefore, by reversal of terms, it follows that DQ:DO = TQ:OT and NU:ND = T′U:T′N. From the equal-angles law, however, it follows that tangent BT bisects angle QBO, and tangent T′F bisects angle NFU. By Euclid, VI.3, then, QB:BO = TQ:OT, and NF:FU = T′N:T′U. Consequently, QB:BO = DQ:DO (which = TQ:OT), and NF:FU (which = T′N:T′U) = ND:DU.

118The point here is illustrated in figure 6.7.32c, p. 142, for points U and O within the plane of H′DQ. That plane cuts arc A′G on the mirror, and within this arc the form of O will reflect to H′ from point B′. Accordingly, Q will be the image of O within this plane. Likewise, the form of U will reflect to H′ from point F′, and its image will be N.

119In other words, since points Z and E lie on arc ZOE, they lie the same distance from the mirror's surface in the plane of circle ABG as does O. Consequently, their forms will reflect from a point on the respective arcs on the circle within their respective planes H′DZ and H′DE that is equivalent to point B′ on arc A′G in figure

6.7.32c, and that point in turn is equivalent to point B on arc AG of the original circle in figure 6.7.32.

[120]All seven manuscripts, as well as the Risner edition, designate the arc that is cut as "UOE," but the sense of the passage clearly indicates that the arc in question is ZOE.

[121]The construction of arc RUF according to the circle centered on M with a radius MU is somewhat puzzling. It would have made more sense to locate M between N and U and then draw the arc with radius MU, because in that case the two intersection points R and F on arc ZOE would have flanked U at equal distances, leaving concave arc RUF uniformly disposed with respect to DQ.

[122]The intent of this final phrase is far from clear. Although all the manuscripts have *eodem modo quo in compositis* (or *incompositis*), Risner alters the reading to *eodem modo quo incompositae*, so that the sentence would translate to: "Moreover, compound misperceptions occur in these mirrors the same way as uncompounded [ones]." I take the intent here to be that, as in the other types of mirrors, so in these types, compound misperceptions arise in the same essential way; see note 92 above for a brief explanation of compound misperceptions.

[123]The list of lines intersecting at O varies among the manuscripts, but it is quite clear from proposition 17, pp. 190-192 above, that lines HA, BQ, and TG intersect at O. Three of the manuscripts (O, L3, and E) include TK in this list, and three of them (F, P1, and S) include EK, as does Risner. The problem is that point K lies on the cylinder's axis below D, and EO, which forms the base of isosceles triangle EBO, lies in the plane of circle BF, which is intersected by the axis at point L. Therefore, line EO lies outside the circle (and thus the axis), and point K lies below the plane that includes TU, TZ, TGO, and EO. Accordingly, TK cannot possibly intersect EO, and EO cannot possible pass through point K on the axis. For those reasons, I have substituted EO for all the readings at that point in the text.

[124]The line of reasoning to this point in the paragraph, as well as for much of the analysis that follows, is puzzling, because CU will never intersect line SI, much less bisect it. This point becomes obvious in light of figure 6.8.33a, p. 145, which represents the situation in figure 6.8.33 from a bird's eye perspective in the top diagram and from a directly facing perspective in the bottom diagram. Let the circle in the top diagram be the top one in figure 6.8.33 that passes through G, which is the point from which the form of T reflects to center of sight E. Let NGZ be normal to point G, and let it intersect the axis of the cylinder at centerpoint Z of the circle. According to the cathetus-rule, then, I, where cathetus TU (which bypasses the axis) intersects the extension of reflected ray EG, will be the image of T from the perspective of E. From the bird's eye perspective of figure 6.8.33a, moreover, points Q and H lie directly below and in line with T, point S lies directly below and in line with I, points B and A lie directly below and in line with G, and points L, D, and K on the axis lie directly below and in line with Z. Accordingly, the form of Q will reach the mirror along line QB directly under and in line with TG and will reflect to E along BE directly under and in line with GE. Since the plane of reflection for Q includes the circle passing through B, the cathetus dropped from Q to the mirror will intersect the axis along line QL directly under and in line with TZ. Q's image C will therefore lie at the intersection of this line

and line of reflection EB. H's image S, finally, will lie directly below and in line with I. Consequently, straight line SI will lie to the left of C, and the entire image SCI will be oriented with its convexity directly facing center of sight E. SI, in the lower diagram, will therefore be bisected at F by the extension of reflected ray EB. Furthermore, it has been established that both catheti TU and HU will intersect at U, as will EO. CU will therefore intersect reflected ray EBCS obliquely, so it cannot possibly intersect SI.

[125]Again, by recourse to figure 6.8.33a, p. 145, it is true that Q lies in the plane of triangle CUE, because that is the plane in which Q's form reflects to E. It is also true that C lies in the triangle CEI, so it follows, as claimed, that C lies on straight line EB. Yet, as we just saw, the claim at the beginning of the paragraph that CU bisects SI is false, because it does not intersect it anywhere. EC does bisect SI, so bisection-point F will lie on that line. Accordingly, from the bird's eye vantage of figure 6.8.33a, it will lie below and directly in line with I.

[126]In other words, since it has already been established that the plane containing HUT bypasses the axis, then, if we assume that the axis somehow intersects line HU at some point, that point will have to be D, where the plane of reflection containing HU intersects the axis. Hence, within that plane, HU will have to both intersect and bypass the axis at the same point.

[127]See Smith, *Alhacen on the Principles*, xxxix-xliv for a summary account of how the number of possible images will vary from one to four depending on the placement of the object-point and center of sight within a great circle on a concave spherical mirror; that account includes citations of specific relevant propositions in book 5, chapter 2. Moreover, on the basis of this analysis, Alhacen goes on in proposition 52, ibid., 478-481, to show that, when the center of sight and the object-point lie in a plane that cuts an elliptical section on the surface of a concave cylindrical mirror—as is the case for both T and H with respect to O—there can be as few as one and as many as four possible images of the object-point.

[128]The possibility of there being multiple images of C depends on the placement of center of sight O. If it lies outside the great circle of reflection, as in figure 6.8.33, then clearly there can be only one image. As to where the image of C might lie on line of reflection OB or its extension along BQ, that depends on C's placement with respect to the mirror's surface.

[129]Presumably this qualification is meant to account for situations in which C is displaced from its original position in such a way that image TQH of object-line SCI is convex or concave within plane TOH. In that case, of course, the convexity or concavity of the resulting image will face the eye directly, so it will be far less clearly perceptible than it would be if viewed from the side.

[130]As Alhacen points out, if we take SCI as the visible line, then it is possible for the points on it to have up to four images, depending on the placement of those points with respect to the center of sight. Thus, as illustrated in figure 6.8.33a, p. 145, the original case based on convex object-line SCI and center of sight O yielded straight-line image TQH. However, depending on SCI's placement with respect to O within the mirror, the three points S, C, and I could have as many as four images, in which case line SCI could have as many as four images—all of them distributed

in various ways with respect to center of sight O (i.e., behind the reflecting surface, as with image TQH; between the center of sight and the reflecting surface; at the center of sight itself; or behind the center of sight). Moreover, if we displace C from its original position while leaving points S and I unchanged, and if we focus on the original images T and H of I and S, then there can be as many as four separate images. Take the simplest case, in which the center of sight and endpoints S and I of the object-line are equidistant from the center of curvature. Thus, as represented in figure 6.8.33b, let C be shifted to C′ so that C′L = OL and so that the circle passing through C′, L, and O intersects the circle of the mirror at points B2 and B3. The form of C′ will therefore reflect from those two points, as well as from B and B1. Accordingly, the images of C′ will be distributed as follows: the image for reflection from B will be C′1, where reflected ray BO intersects cathetus C′L behind O; the image for reflection from B1 will be C′2, where cathetus C′L intersects reflected ray B1O; the image for reflection from B2 will be C′3, where cathetus C′L intersects reflected ray B2O; and the image for reflection from B3 will be C′4, where cathetus C′L intersects reflected ray B3O. All of the resulting images are magnified insofar as cross-section TH common to all of them is longer than cross-section SI of object-line SC′I. However, SC′I can have as many as sixteen images when S and I are posed to reflect from four points on their respective great circles, and when the four reflection-points are equivalently situated within those respective great circles.

[131]Point K in this proposition corresponds to point Q in proposition 19. In paragraph 5.34 of that proposition (p. 194 above), Q is located where the plane containing E and the cylinder's axis—i.e., plane EX′D in figure 6.8.34a—intersects TH. Since the remainder of the proof from this point applies to that plane, I have supplied a view of it from the side in the diagram at the bottom of the figure in order to make the steps of the construction and analysis easier to follow.

[132]The reason for establishing that MIA is acute is to establish that any line from M that intersects AN below N, and thus below line ENL, will form an acute angle with AN. Consequently, any line drawn from that point to L will form an acute angle with AN on the opposite side. From this it follows that there must be some point on AN below line ENL from which the form of M will reflect to L at equal angles.

[133]That is, since angles FSQ and FMQ are equal, their respective alternate angles LFX and MFX will be equal.

[134]In order to make the construction and analysis that follows easier to visualize, I have represented the ellipse ABG in the middle diagram of figure 6.8.35, since the relevant lines and angles are all formed within its plane.

[135]The proposition to which Alhacen is referring is proposition 33, pp. 221-224 above. More appropriate to the point Alhacen is making here, however, is the analysis of concave cylindrical mirrors in book 4, chapter 5, paragraph 5.56 (Smith, *Alhacen on the Principles*, 344), where he shows descriptively that for any plane of reflection forming an ellipse on a concave cylindrical mirror there are only two possible points of reflection, and they lie at the endpoints of the ellipse's minor axis, which is the common section of the ellipse and the circle passing through those points.

[136]Alhacen's point here is that the reflected ray DG is parallel to the cathetus BL dropped from the point-object L to the mirror, so there is no intersection of the two lines and, therefore, no definite image-location. In paragraph 2.312 of book 5 (Smith, *Alhacen on the Principles*, 448), Alhacen claims that in such cases the image appears at the point of reflection, which in this case is G.

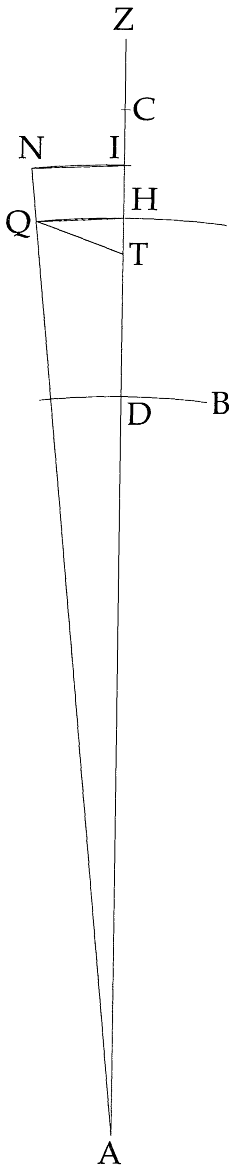
[137]The reference here is to book 5, propositions 39 and 40 (Smith, *Alhacen on the Principles*, 458-460), where Alhacen demonstrates not only that the form of K will reflect to D from a point on arc RC, and only arc RC, but also that there can be only one such point of reflection.

[138]In other words, just as Alhacen reversed the analysis in proposition 19 for proposition 33, here he is reversing the analysis in proposition 22. In that proposition he establishes that the form of line AON, in figure 6.9.37, p. 152, reflects from line of longitude AZE to center of sight R, which faces the convex surface of the mirror, and yields image APY, which is a curved line inside the mirror. Accordingly, the image of A at the cone's vertex coincides with A itself. The form of O, meanwhile, reaches the mirror along incident ray OZ, reflects to center of sight R along ray ZR, and appears at P, where reflected ray RZP intersects cathetus OPH inside the mirror. And the form of N reaches the mirror along incident ray NE, reflects to center of sight R along ray ER, and appears at Y, where reflected ray REY intersects cathetus DYN inside the mirror. Therefore, if we shift the center of sight from R to F and let curved line APY be the visible object, the form of that line will reflect to F from line of longitude AZE and will yield straight line ANO as its image. Accordingly, the image of A coincides with A itself. The form of P, for its part, reaches the mirror along incident ray PZ, reflects to center of sight F along ray ZF, and appears at O, where reflected ray FZO intersects cathetus HPO. And the form of Y reaches the mirror along incident ray YE and is reflected along ray EF, so its image will lie at N, where line of reflection FEN intersects cathetus DYN.

[139]Whether its image is straight, convex, or concave will of course depend on the curvature of AY, which in turn depends on where P lies within plane APY.

[140]In other words, if the center of sight could see those objects face-on, their right-hand sides would correspond to the right-hand side of the image that faces the eye.

FIGURES FOR INTRODUCTION AND LATIN TEXT



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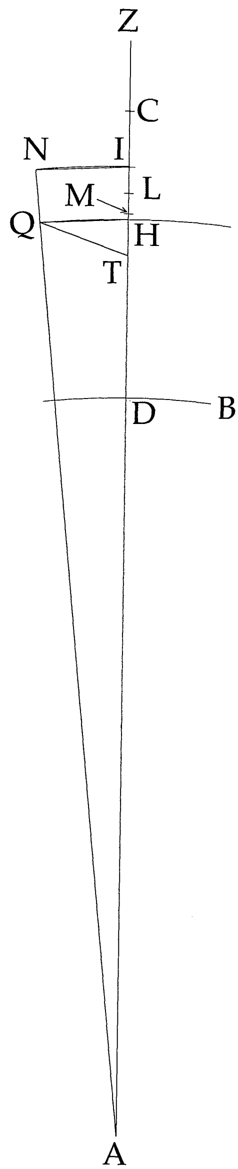

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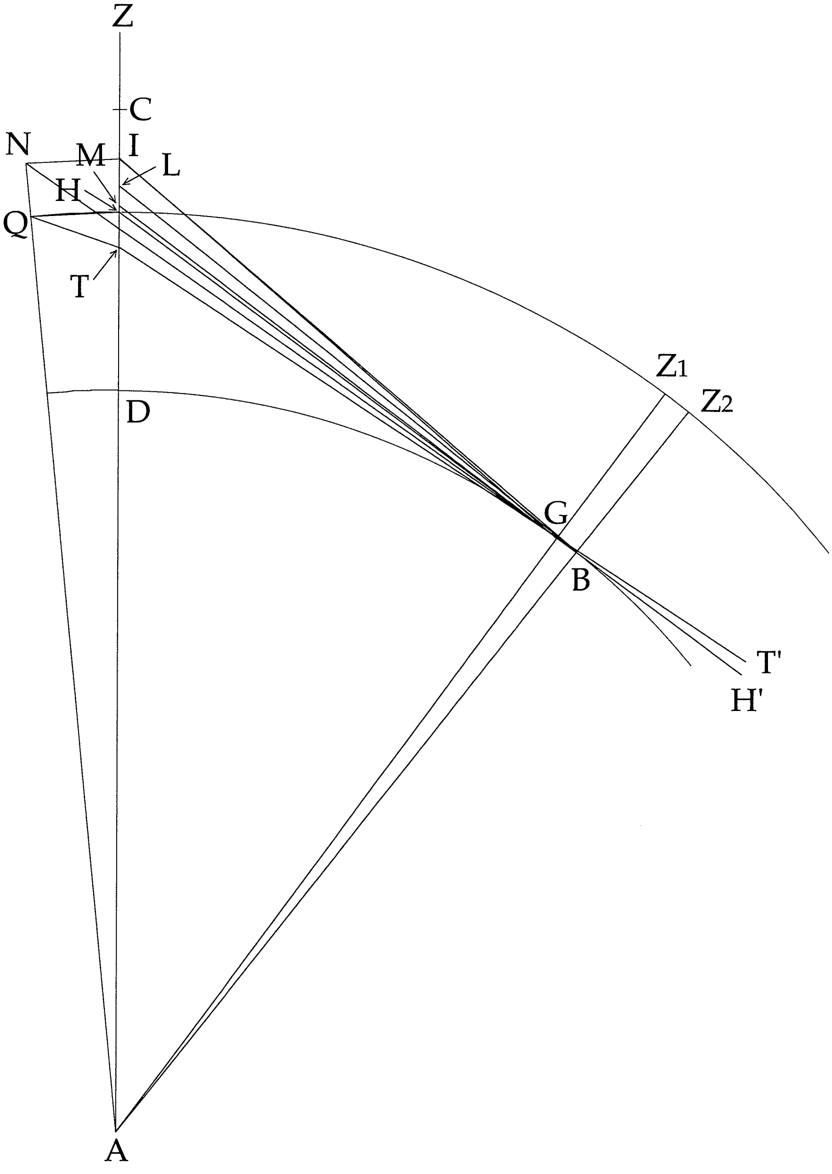

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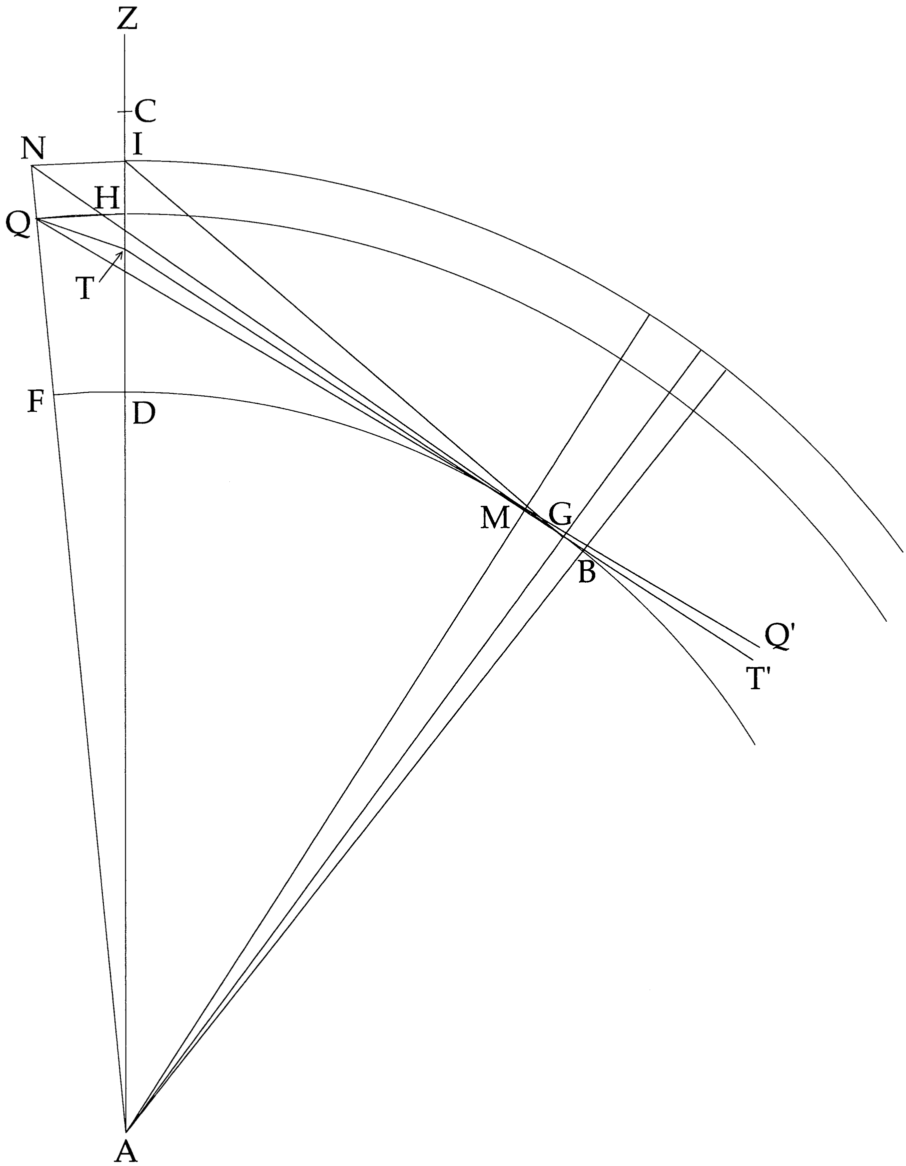

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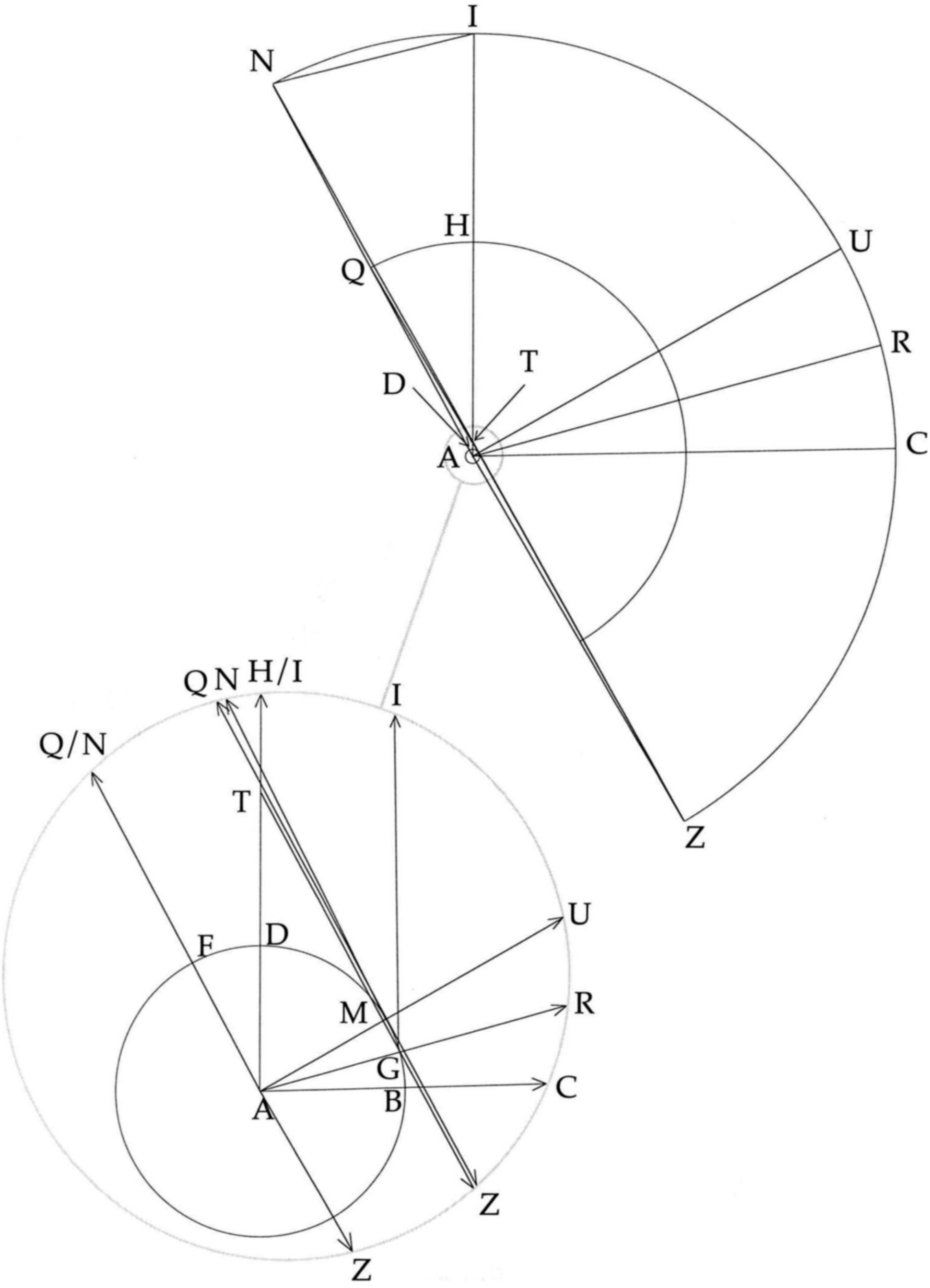

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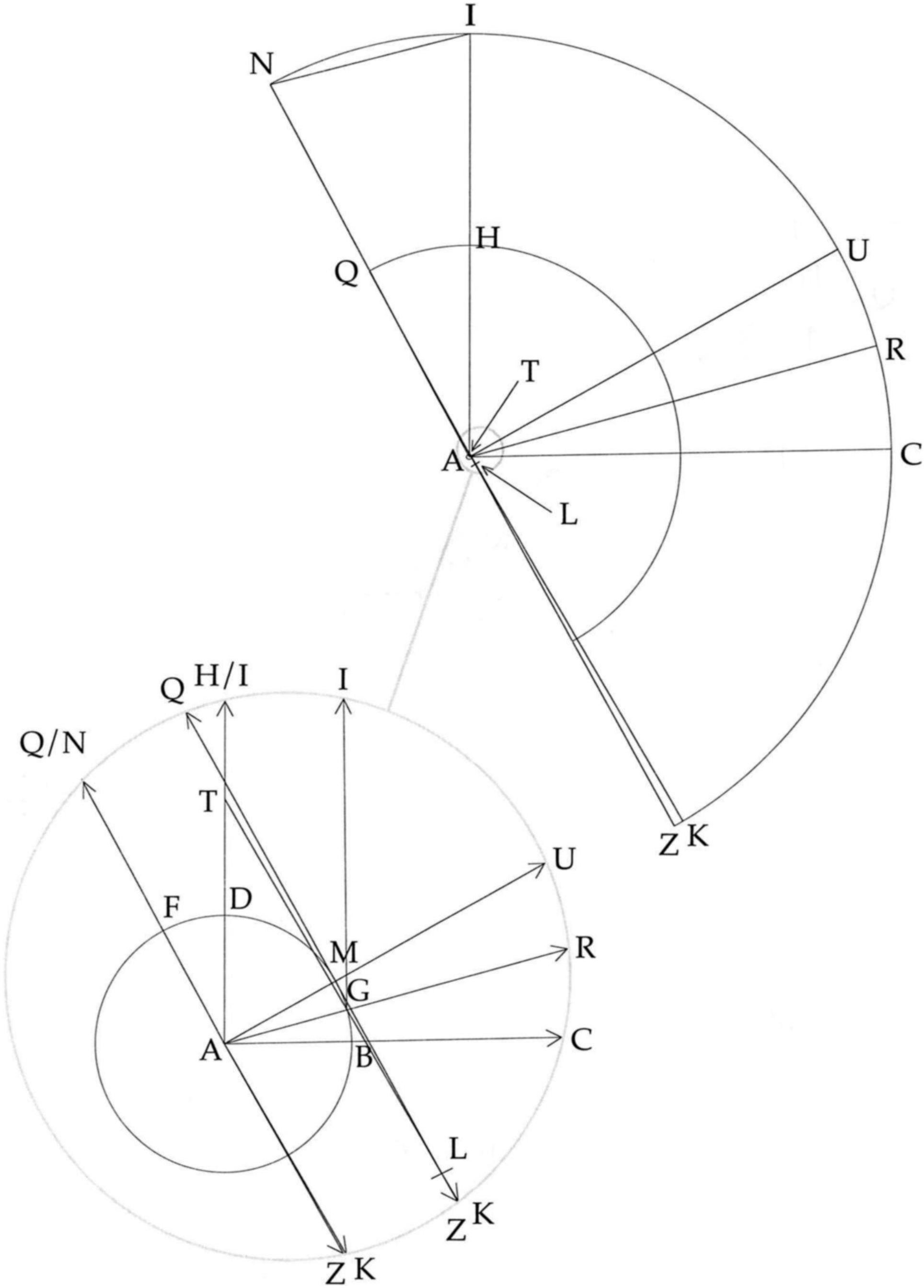

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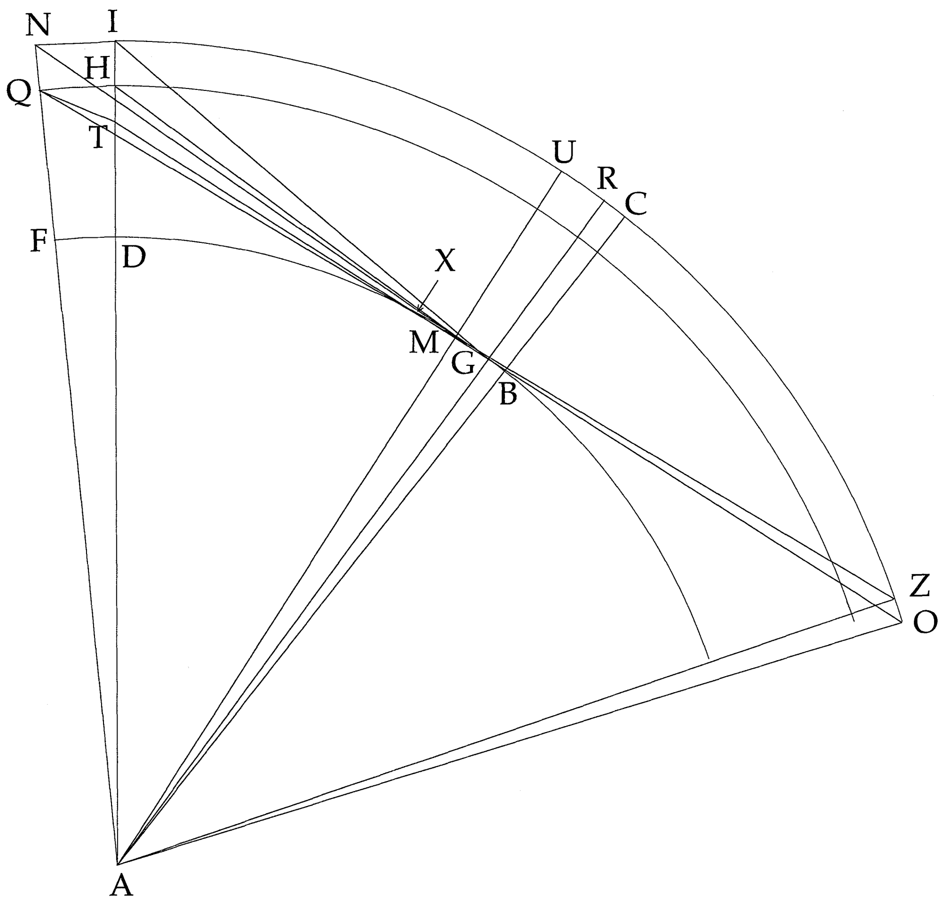

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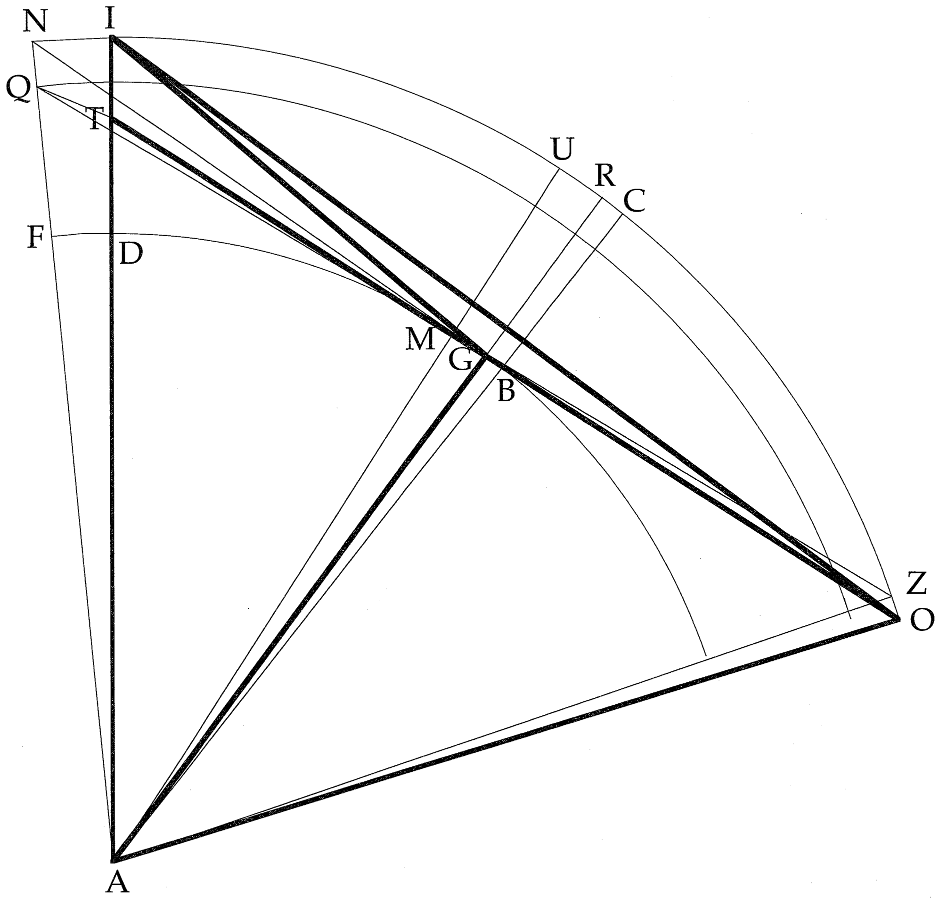

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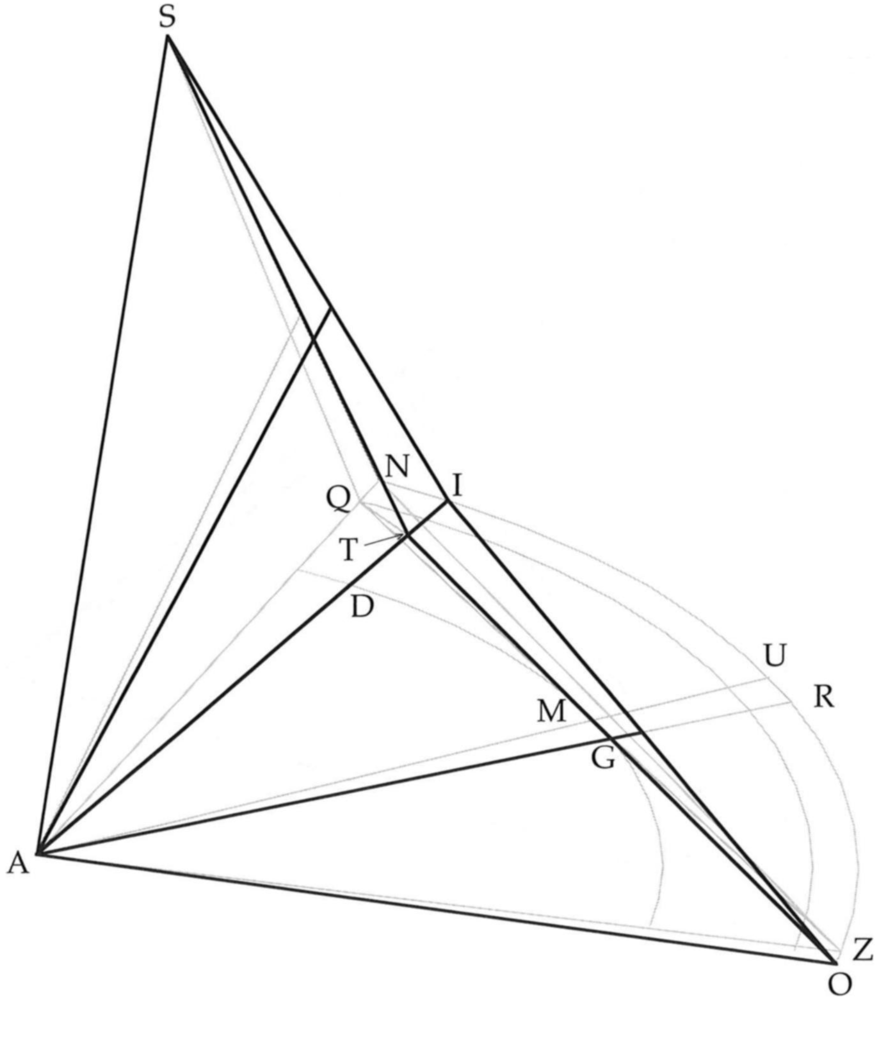

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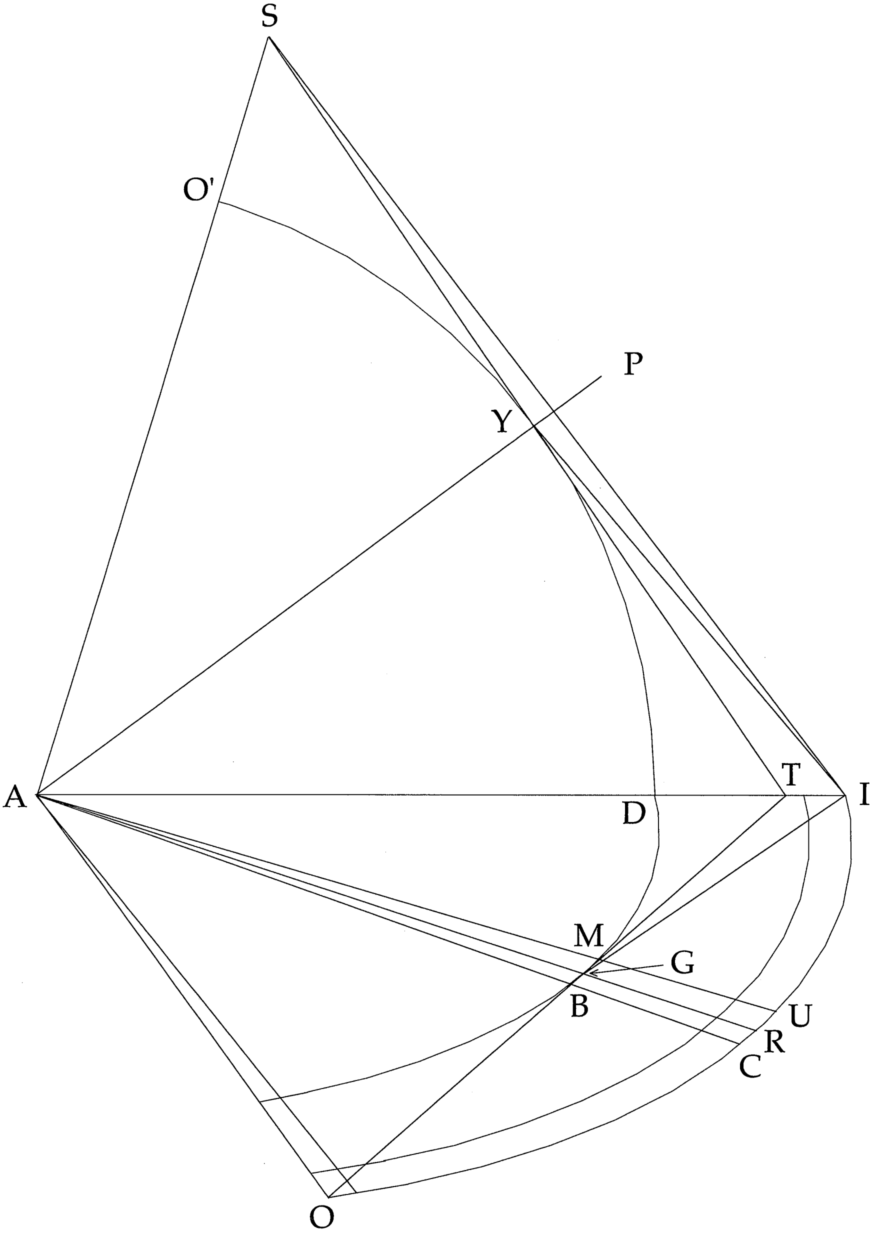

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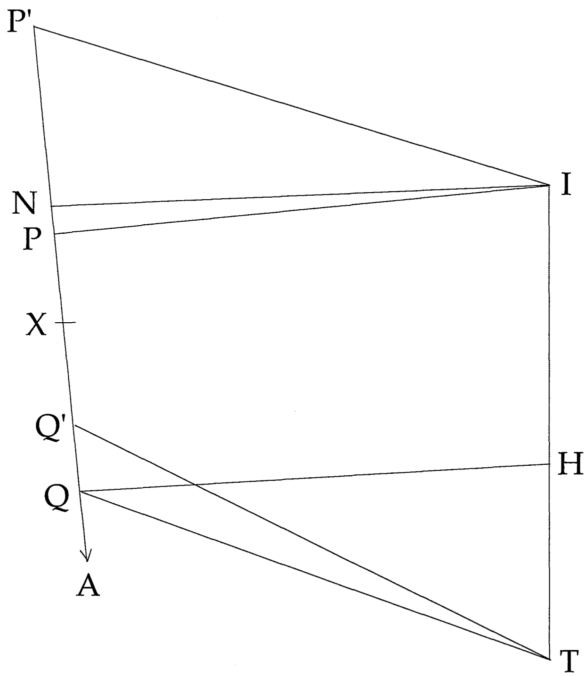

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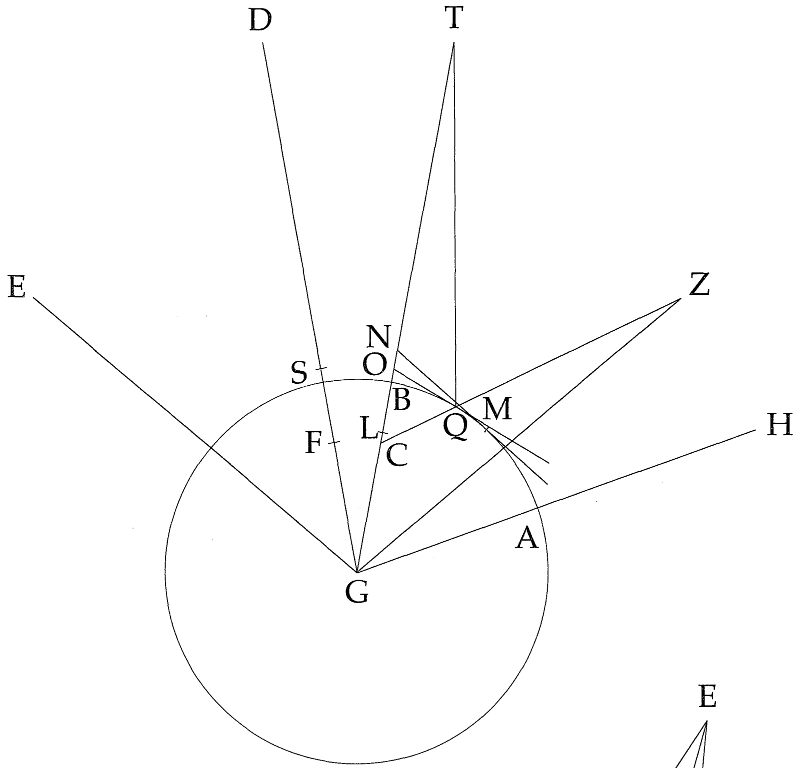

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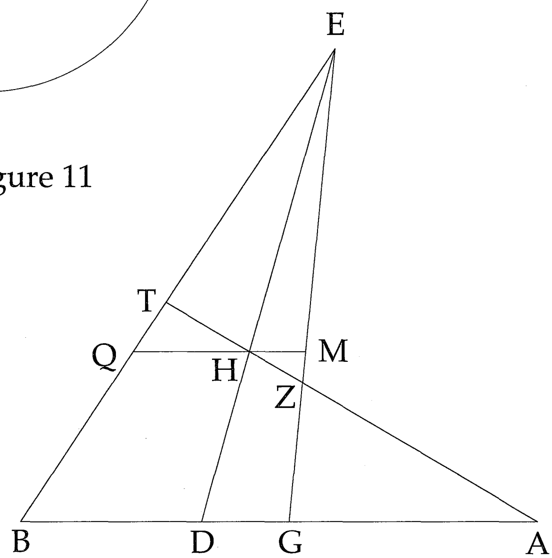

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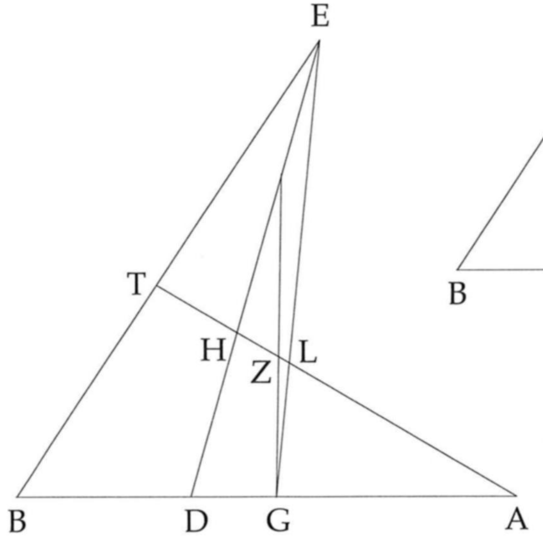

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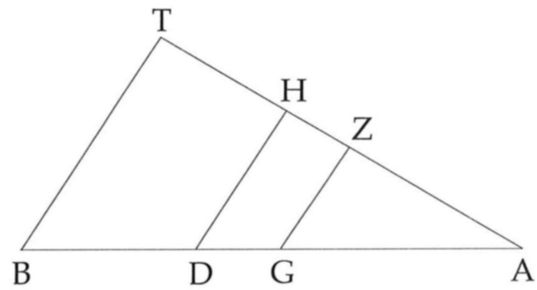

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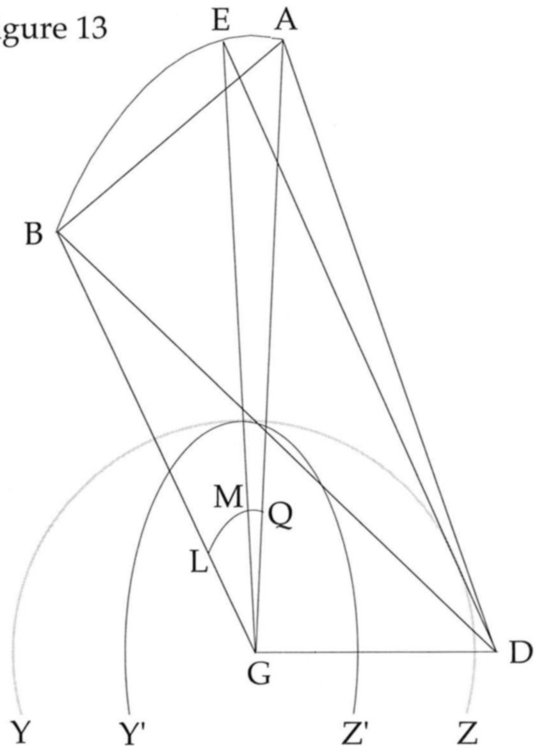

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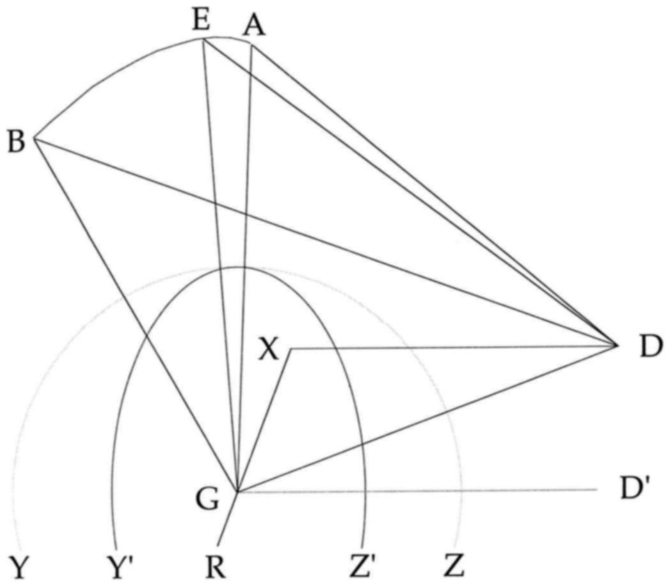

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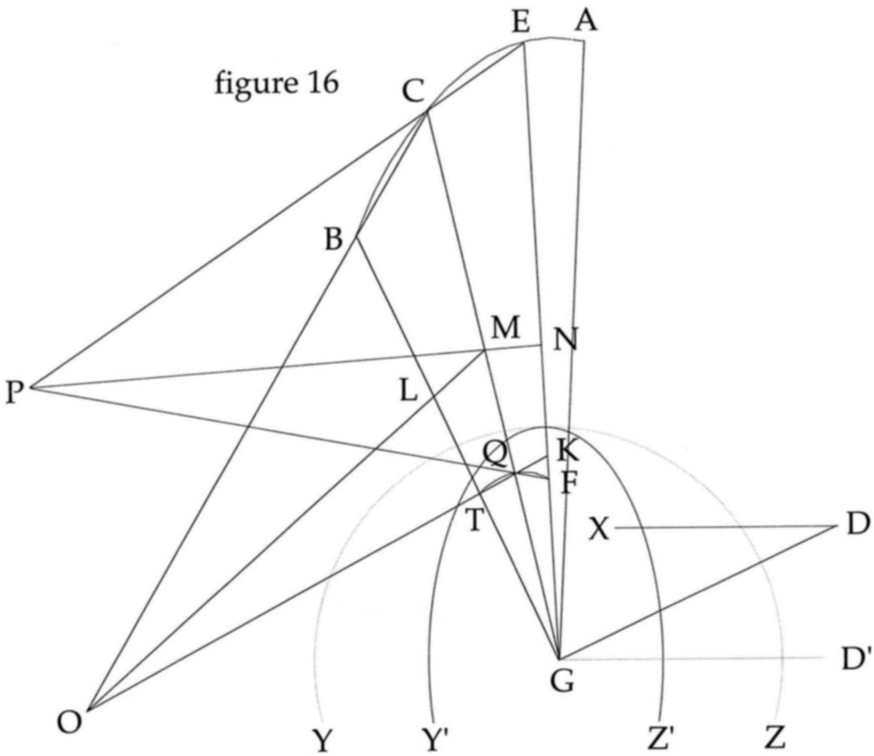

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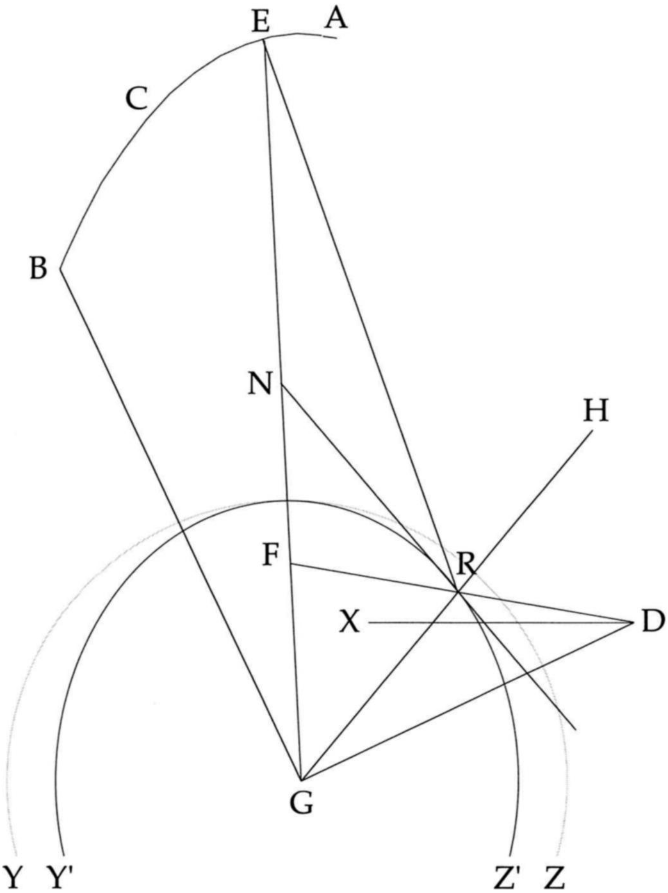

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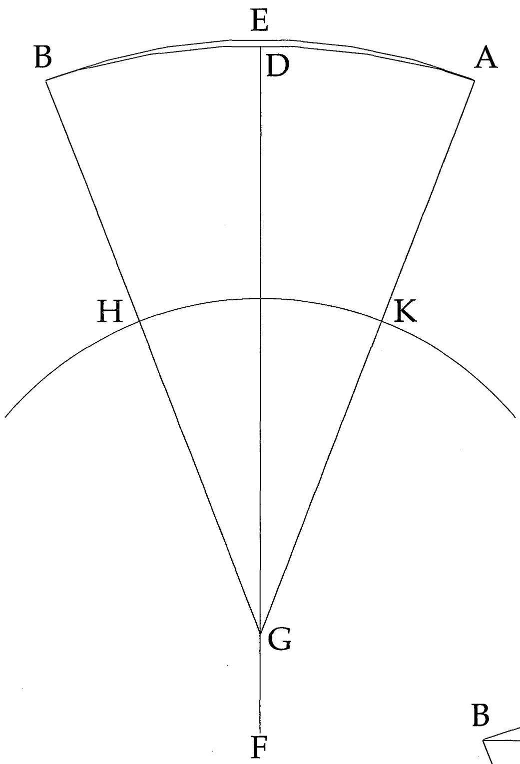

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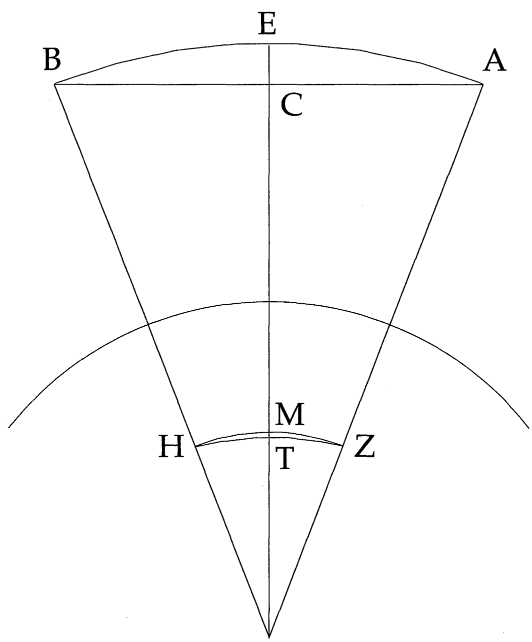

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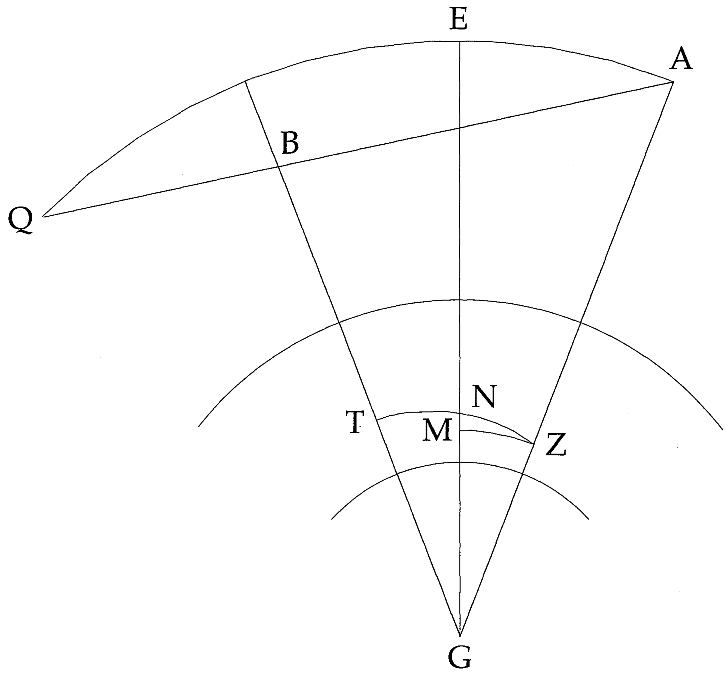

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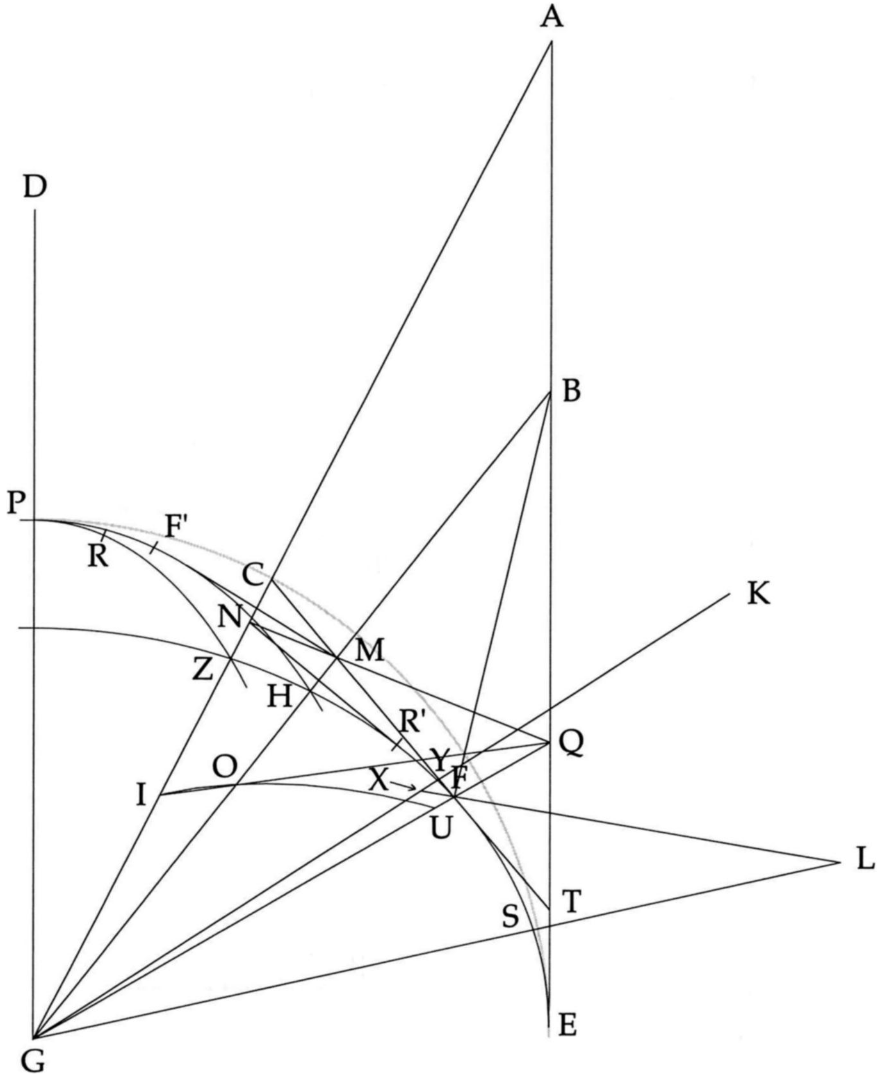

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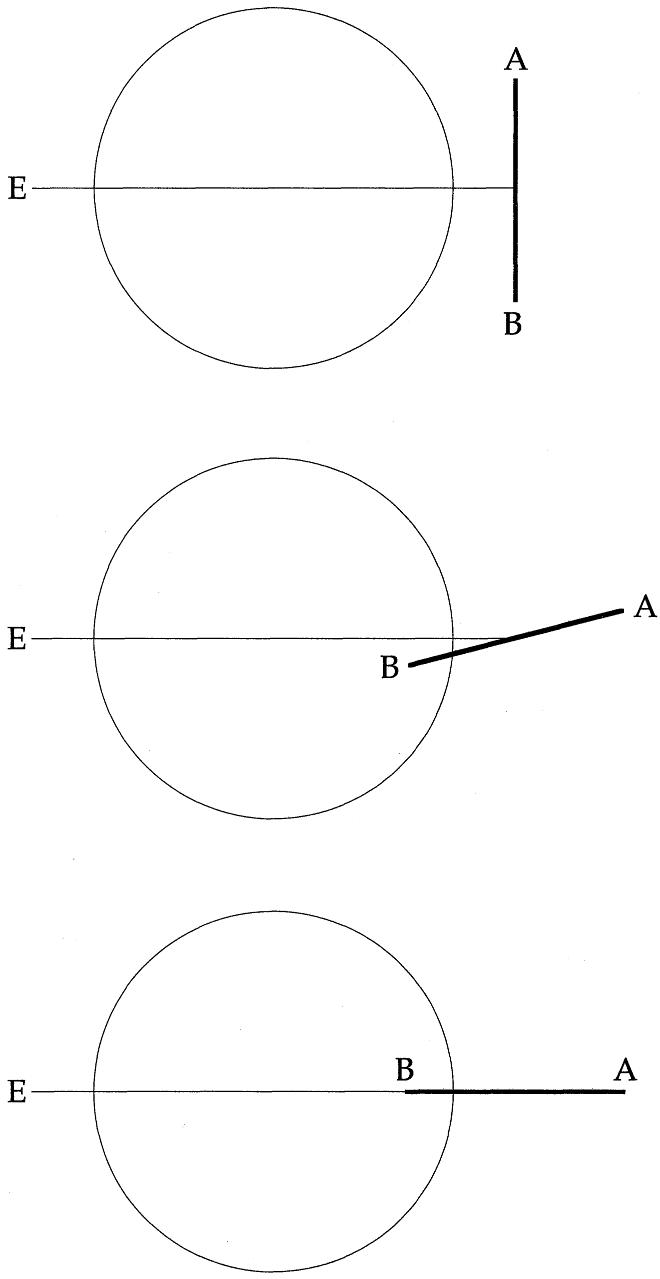

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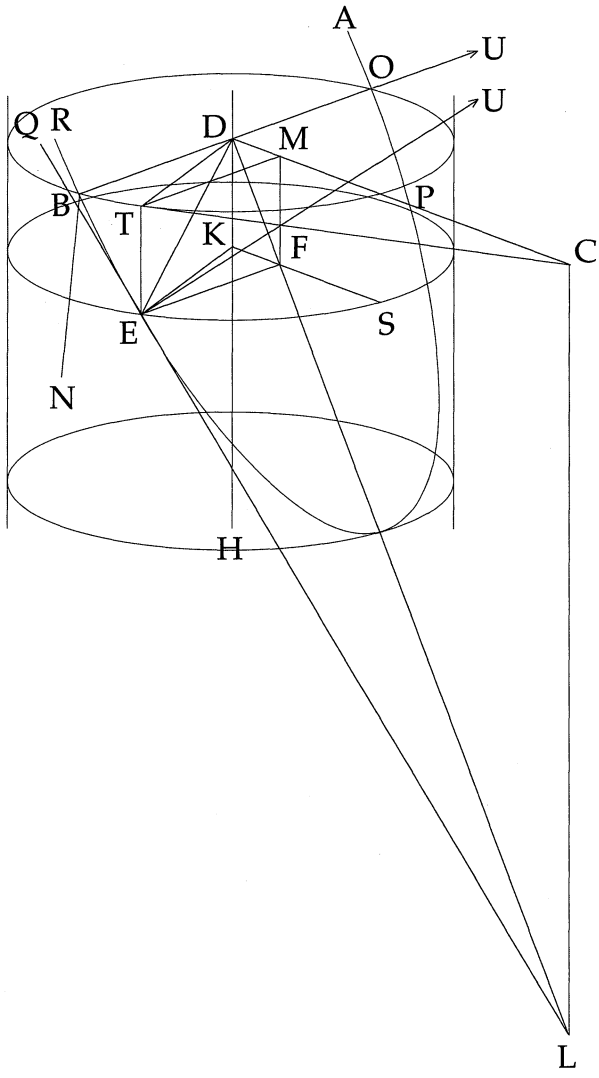

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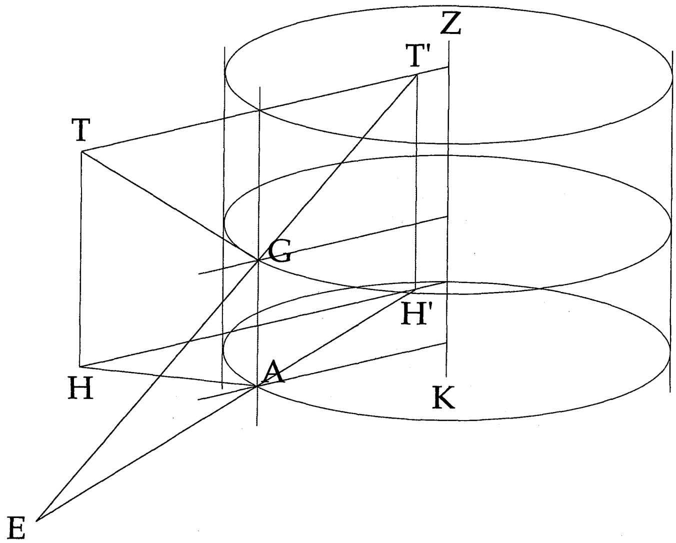

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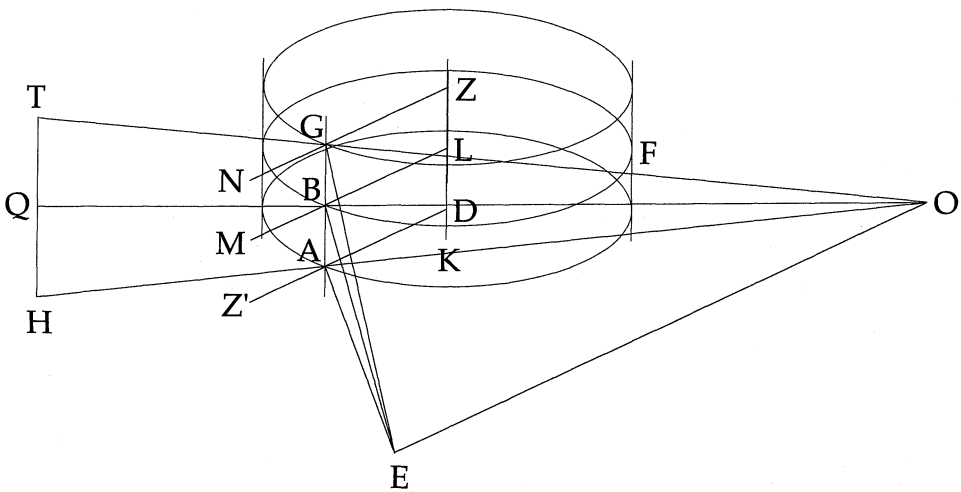

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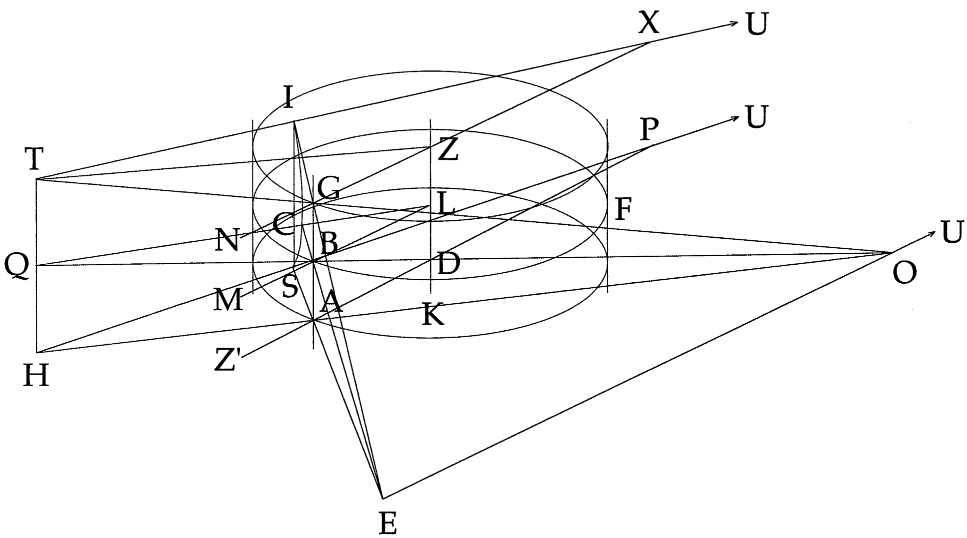

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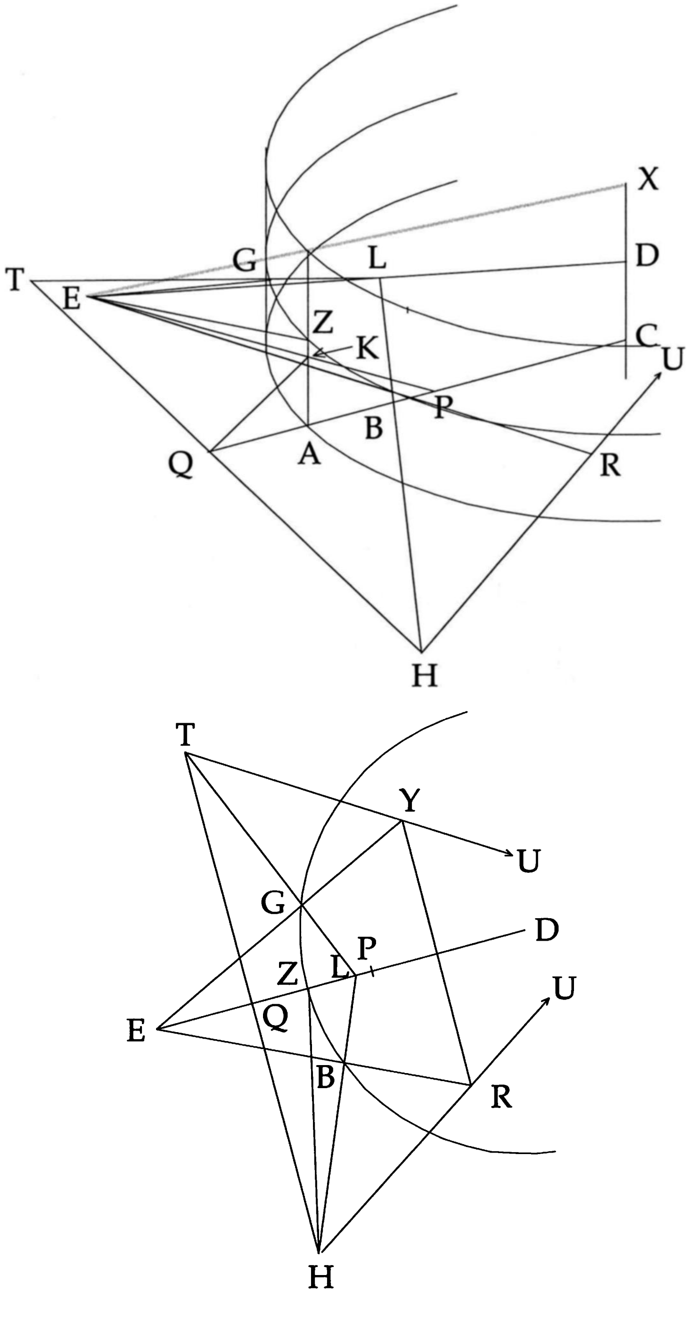

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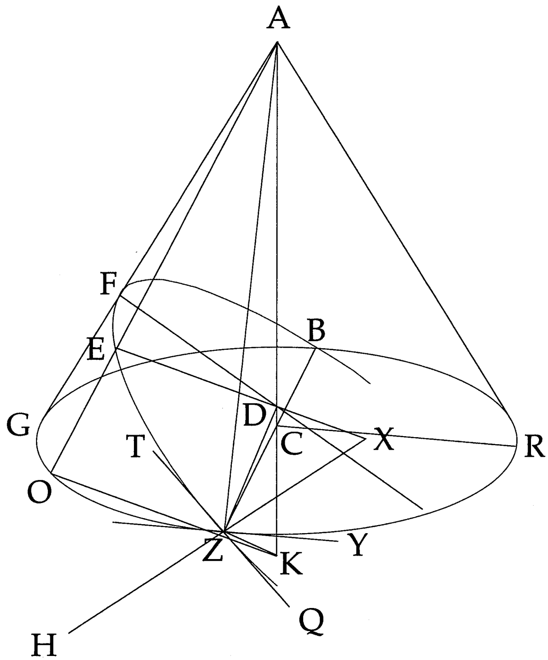

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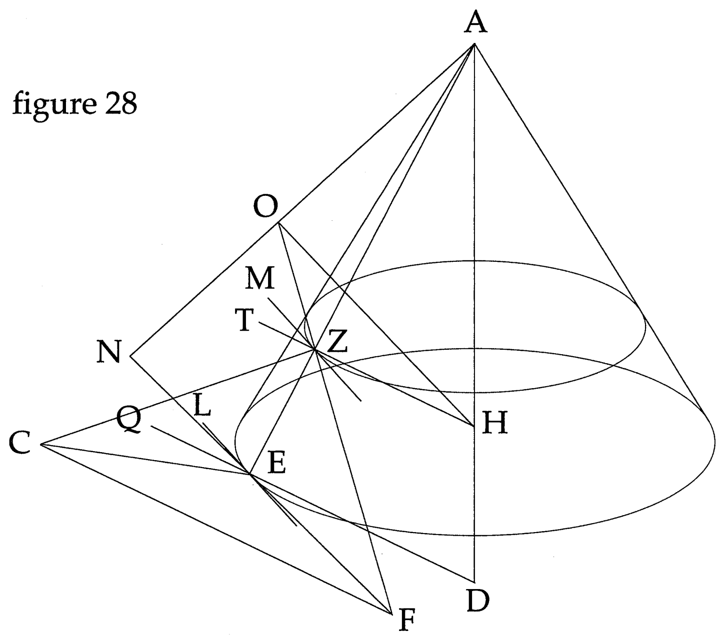

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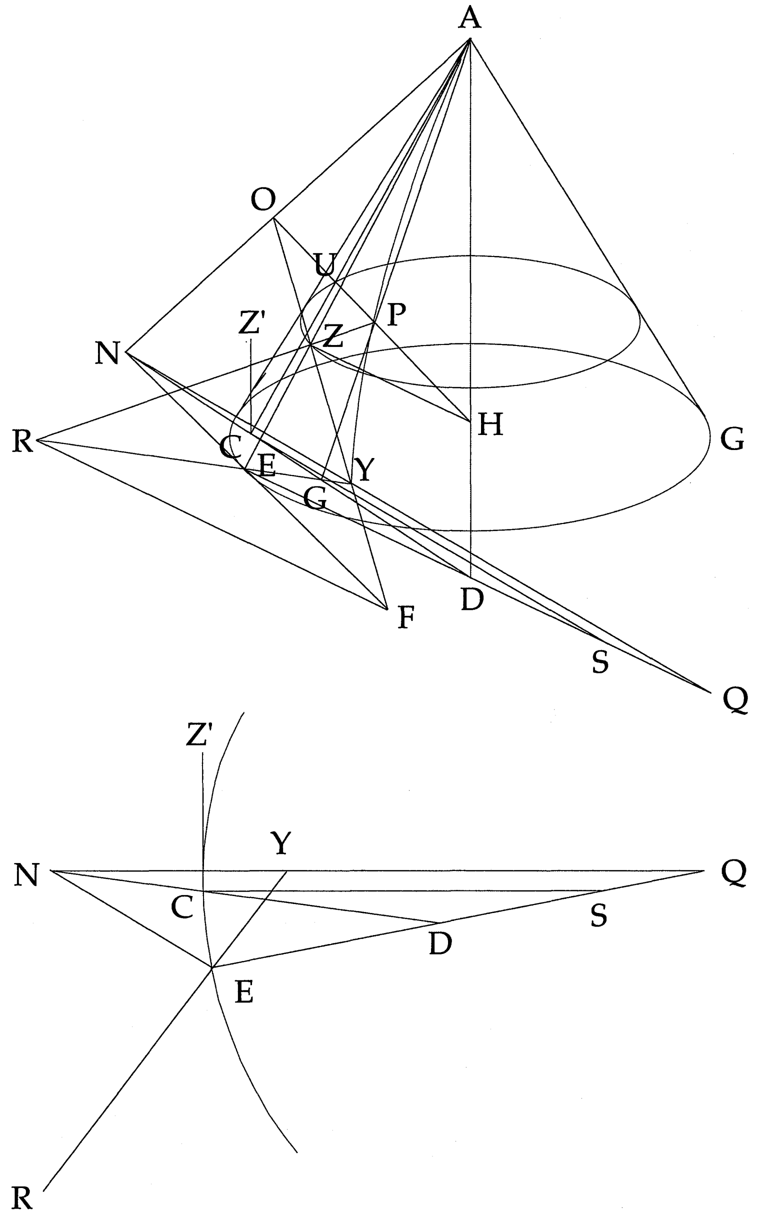

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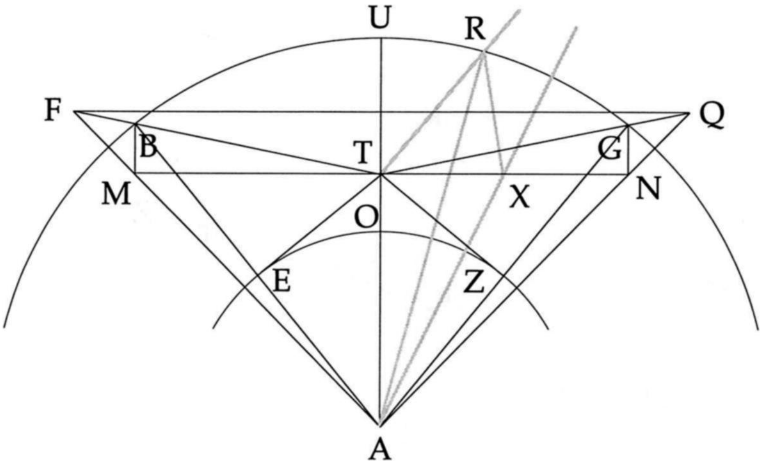

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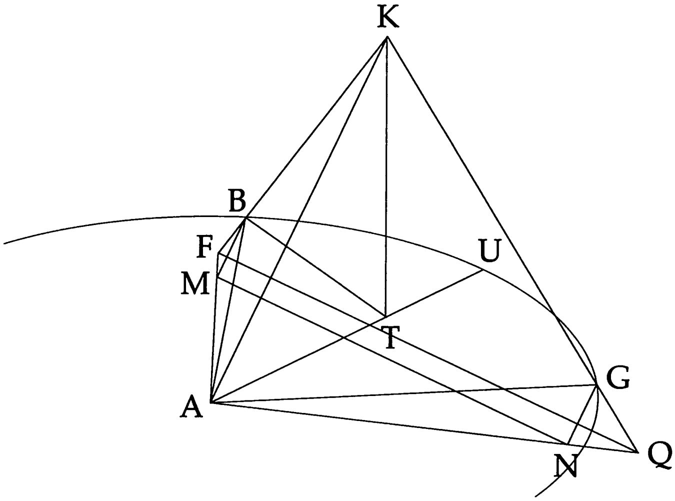

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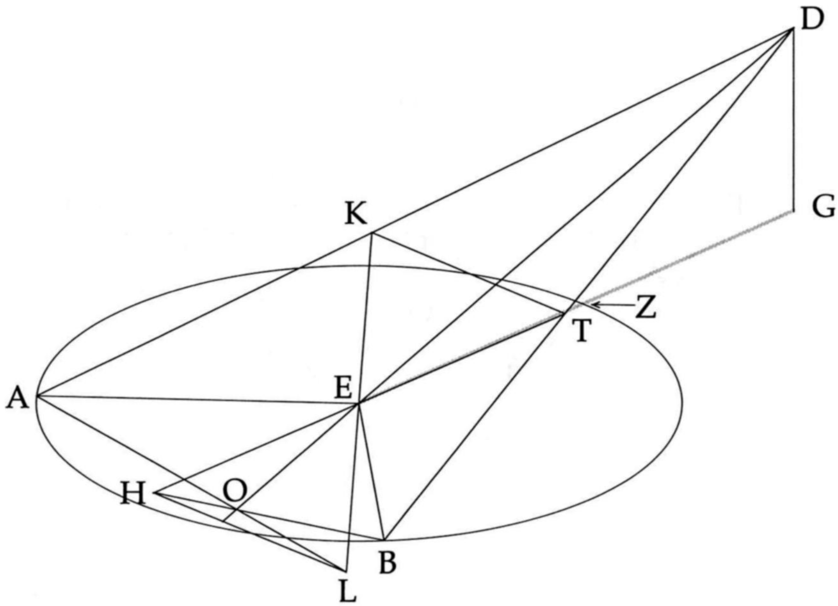

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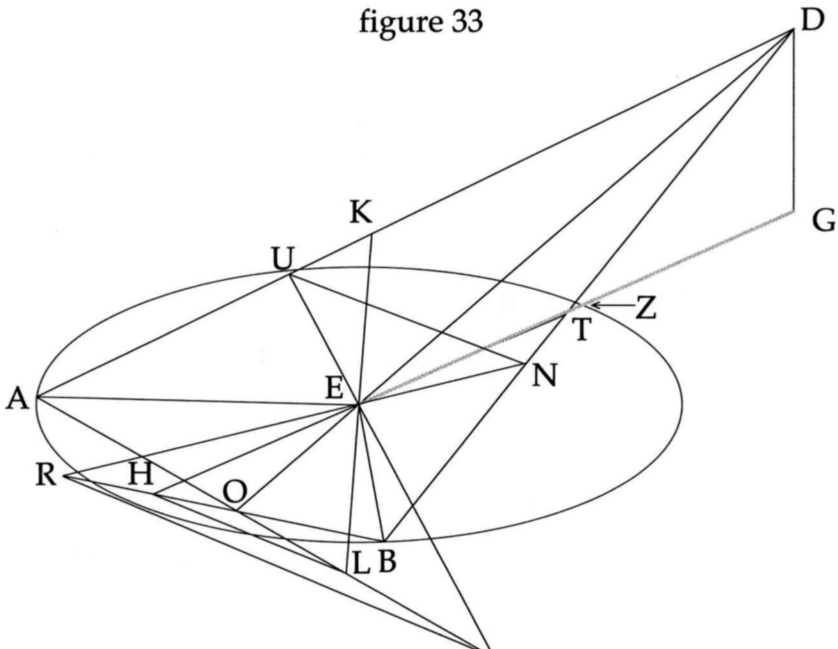

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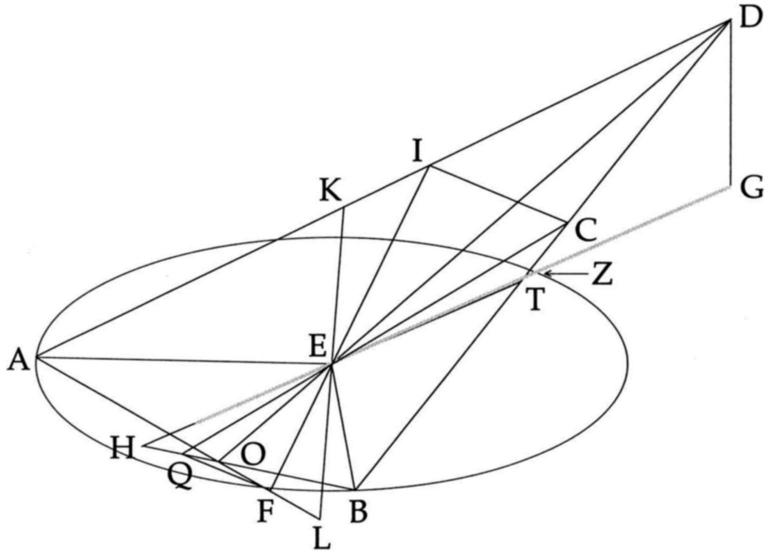

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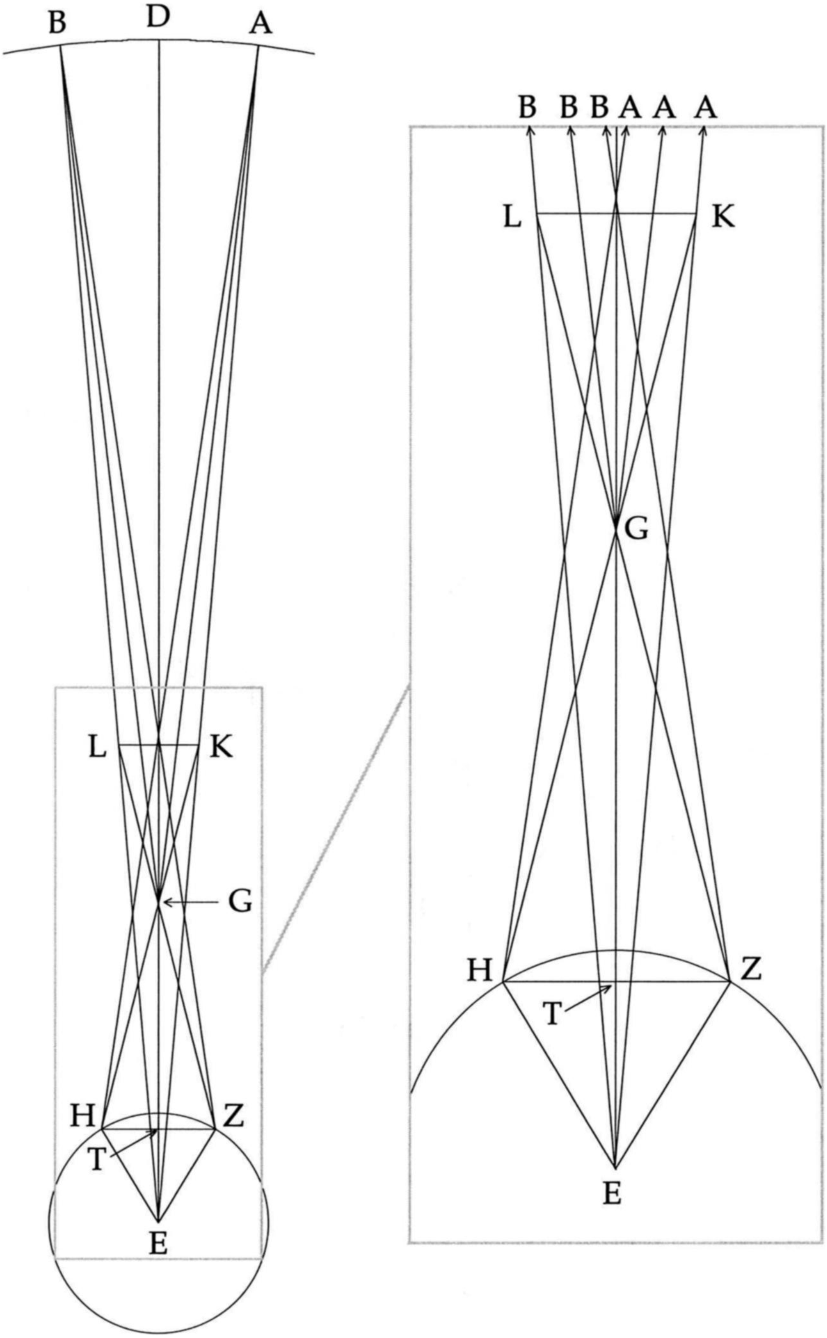

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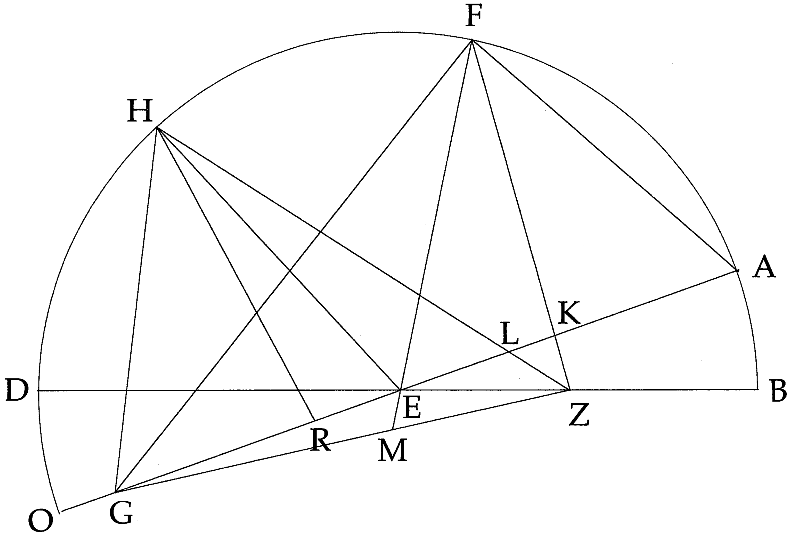

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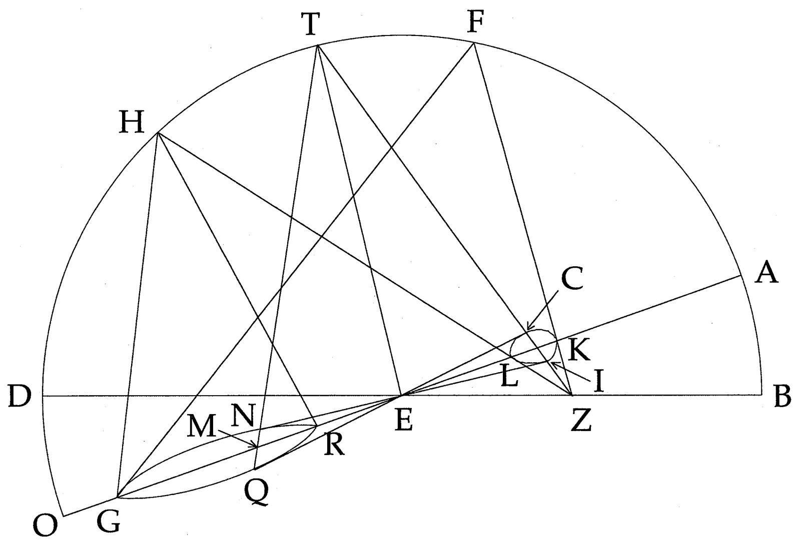

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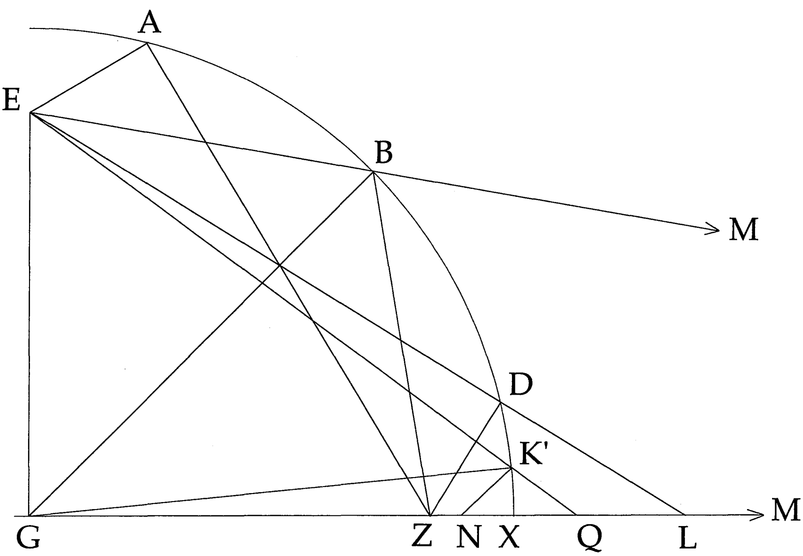

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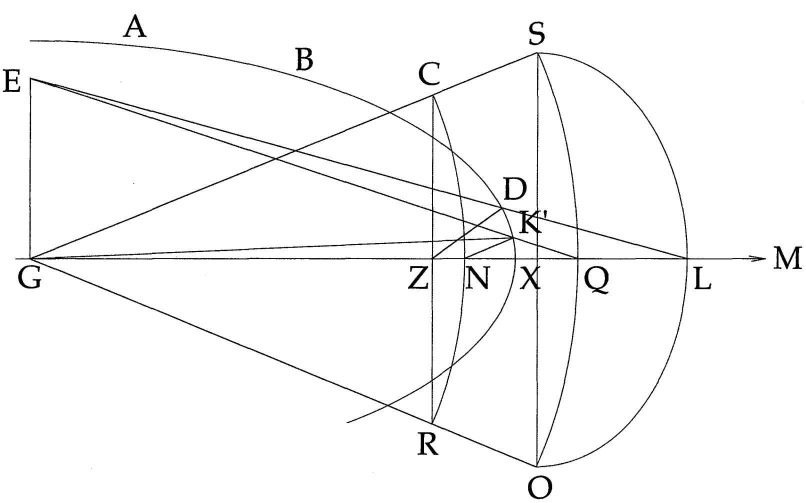

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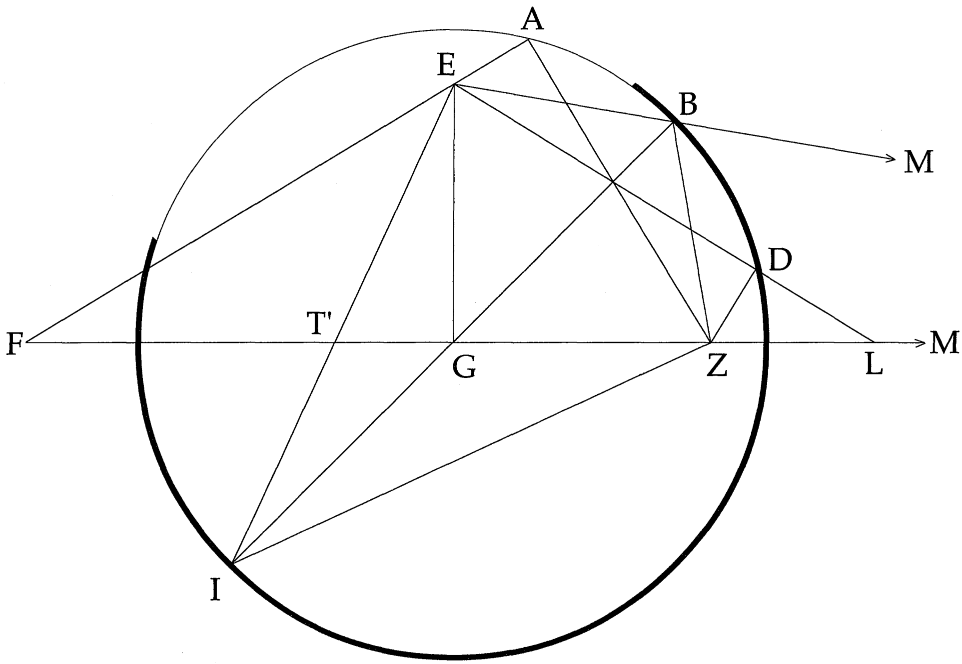

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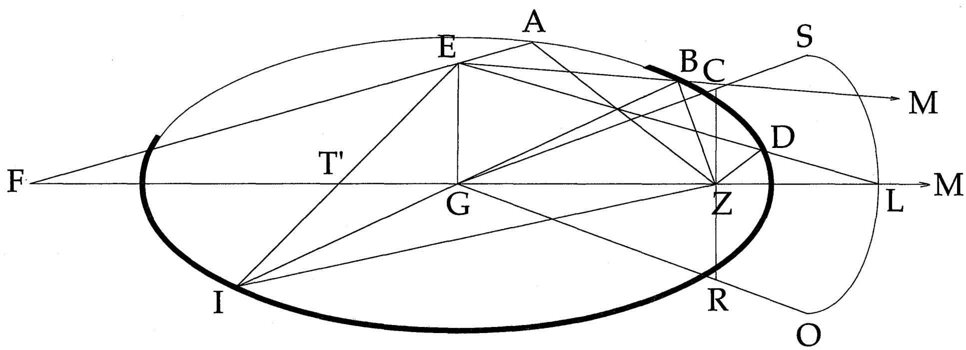

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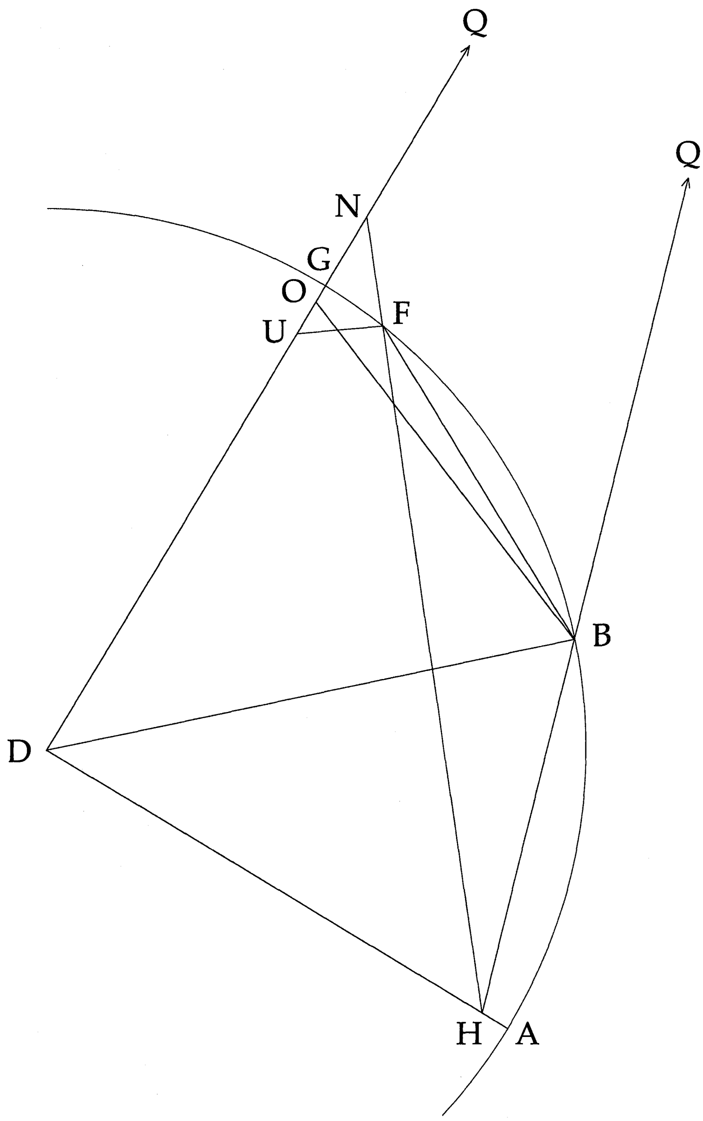

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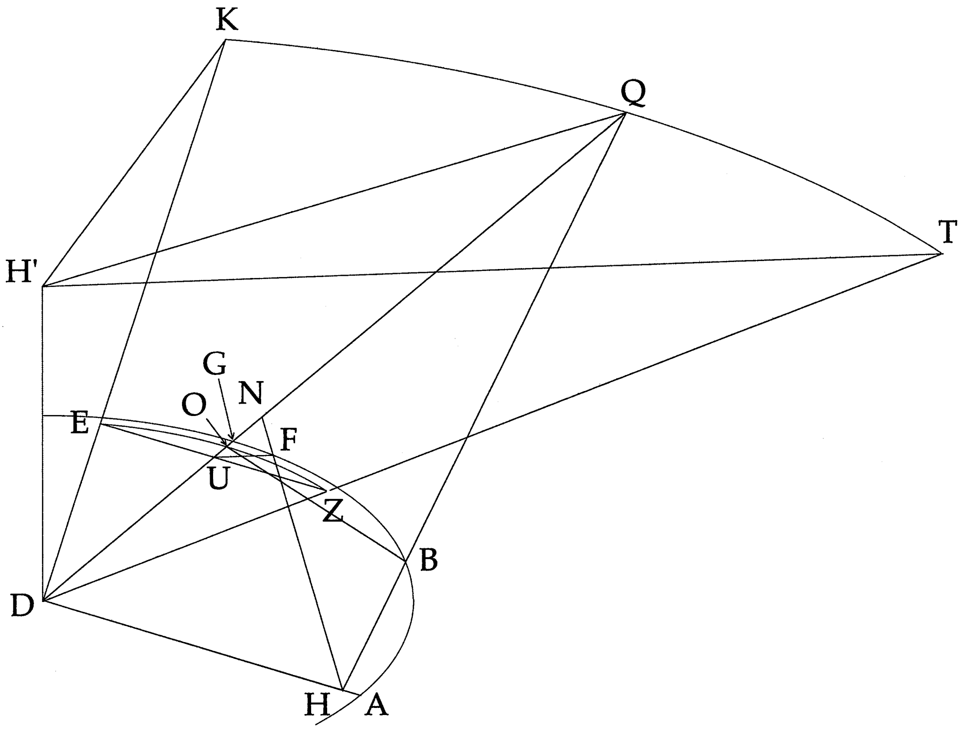

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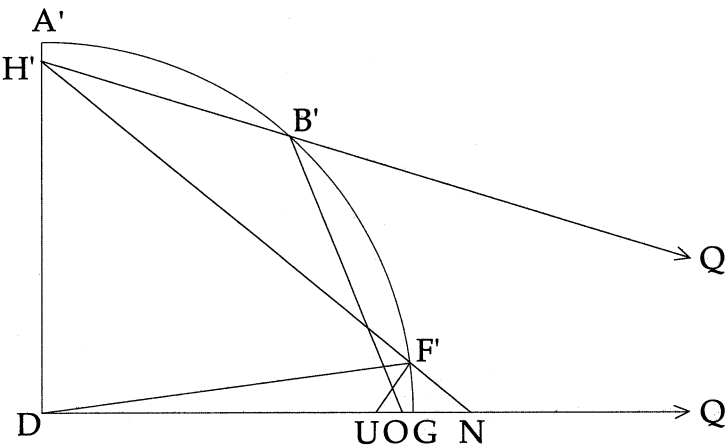

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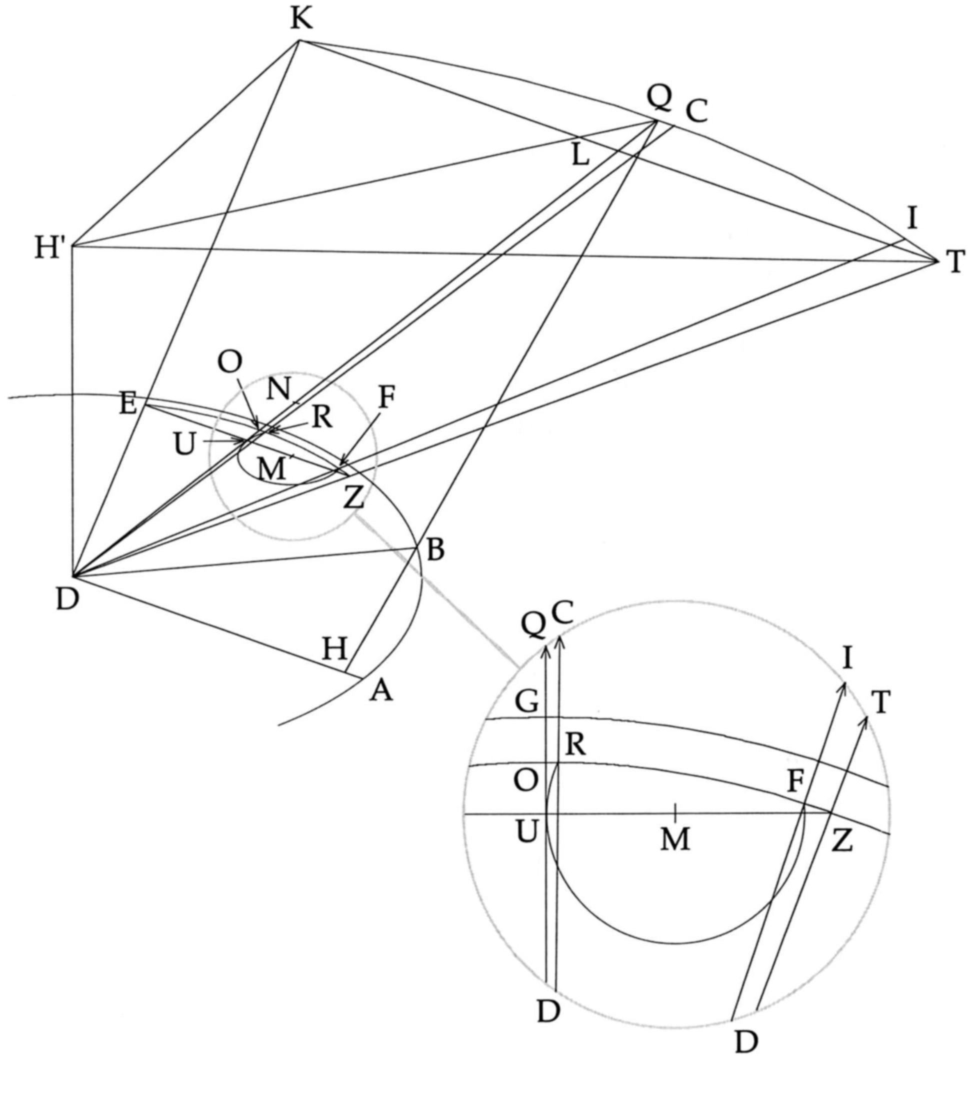

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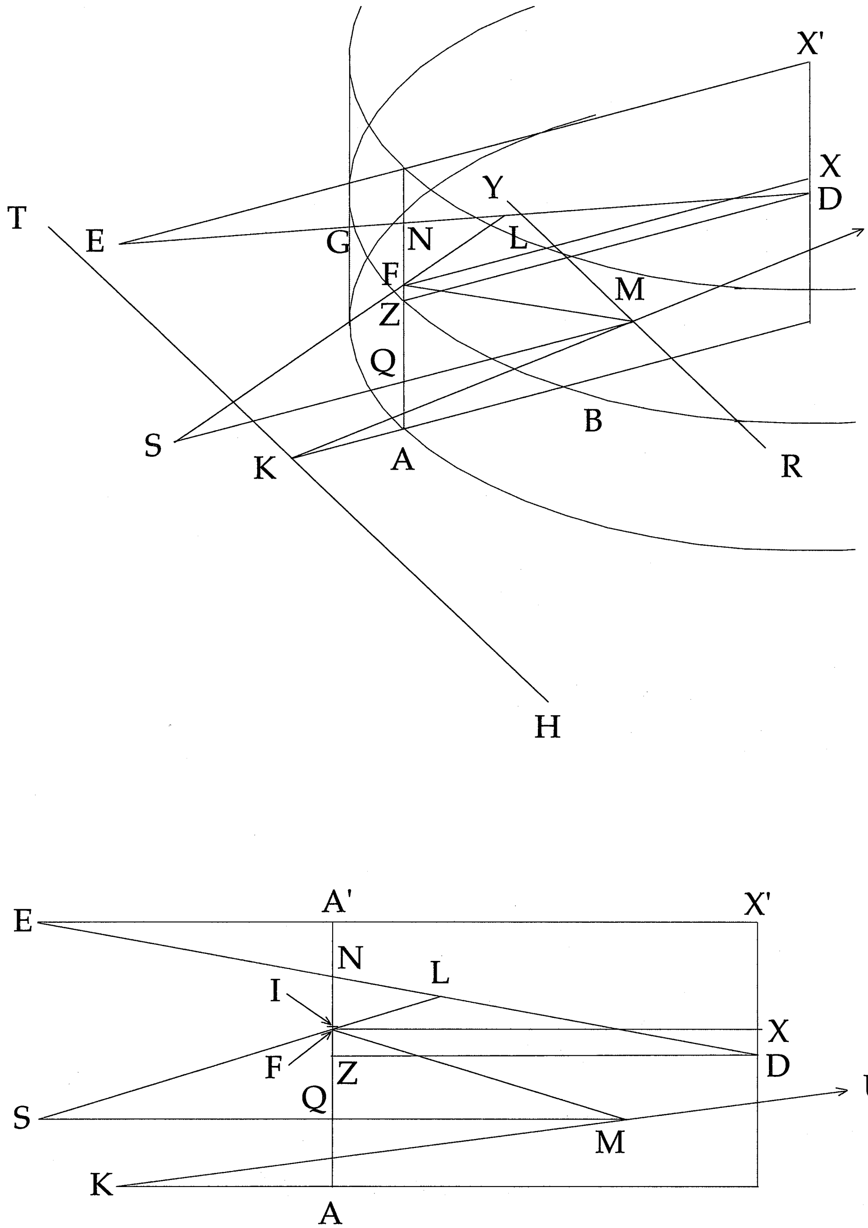

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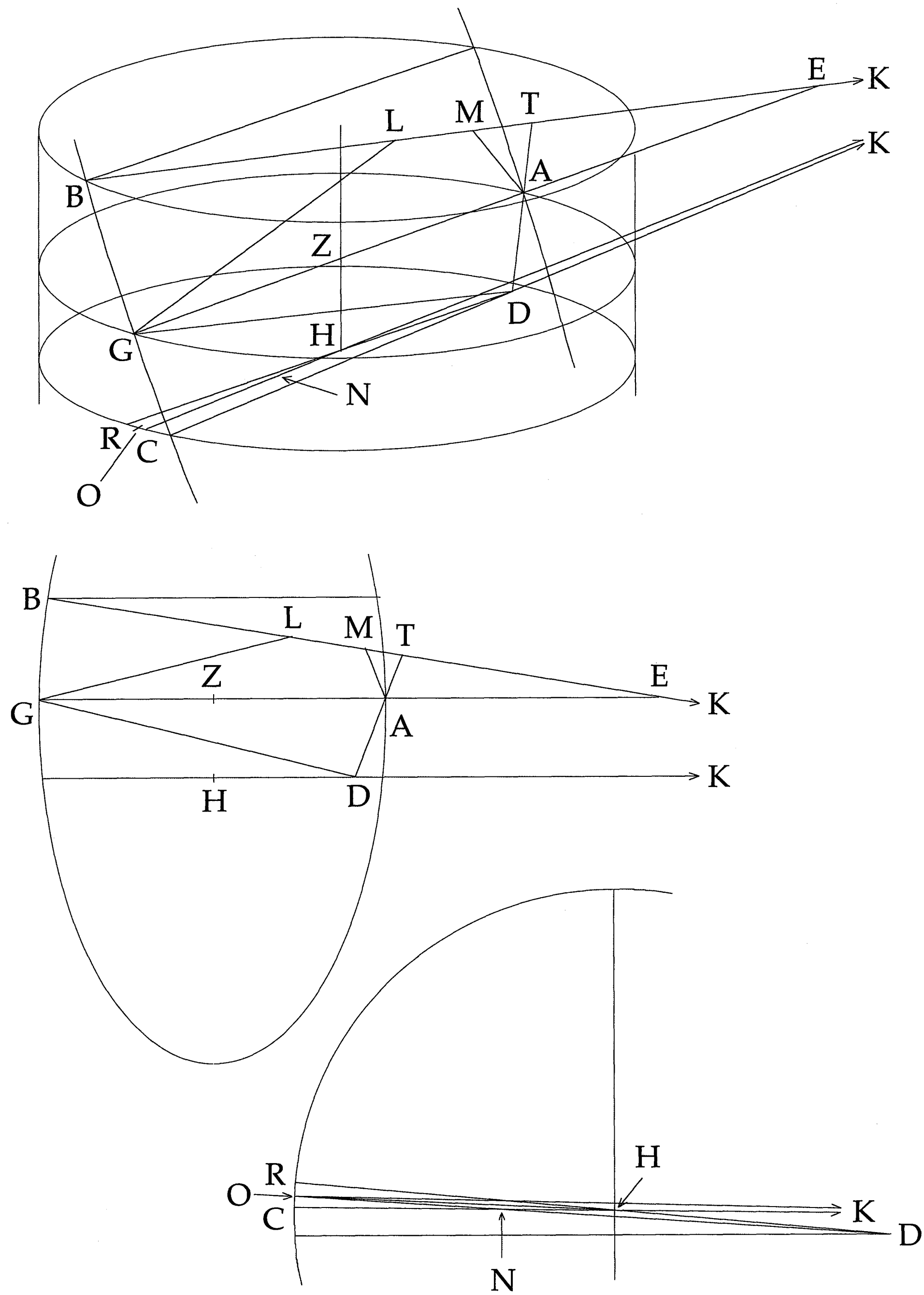

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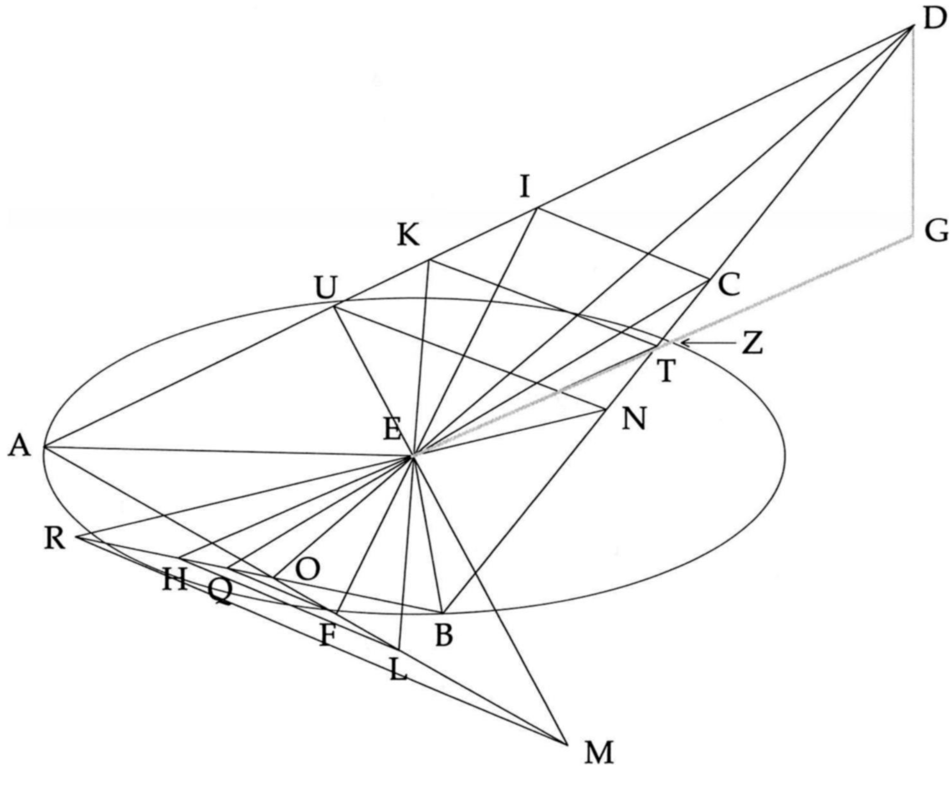

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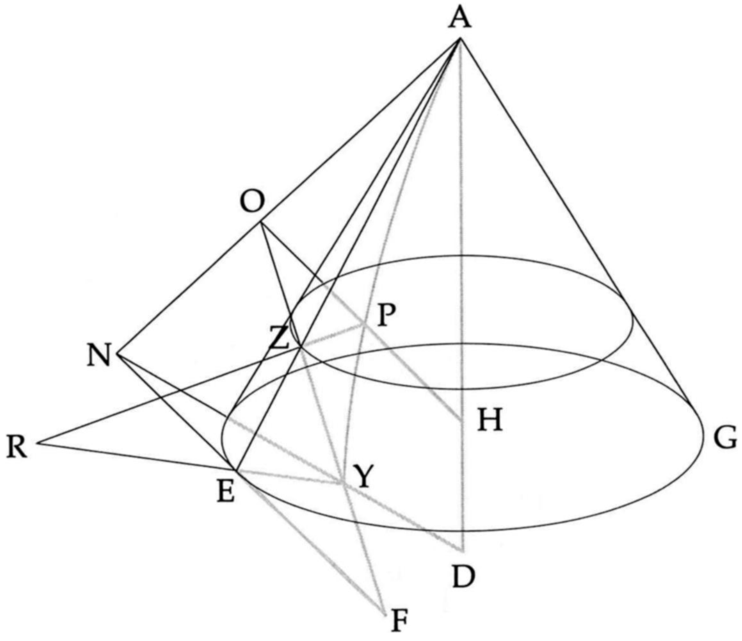

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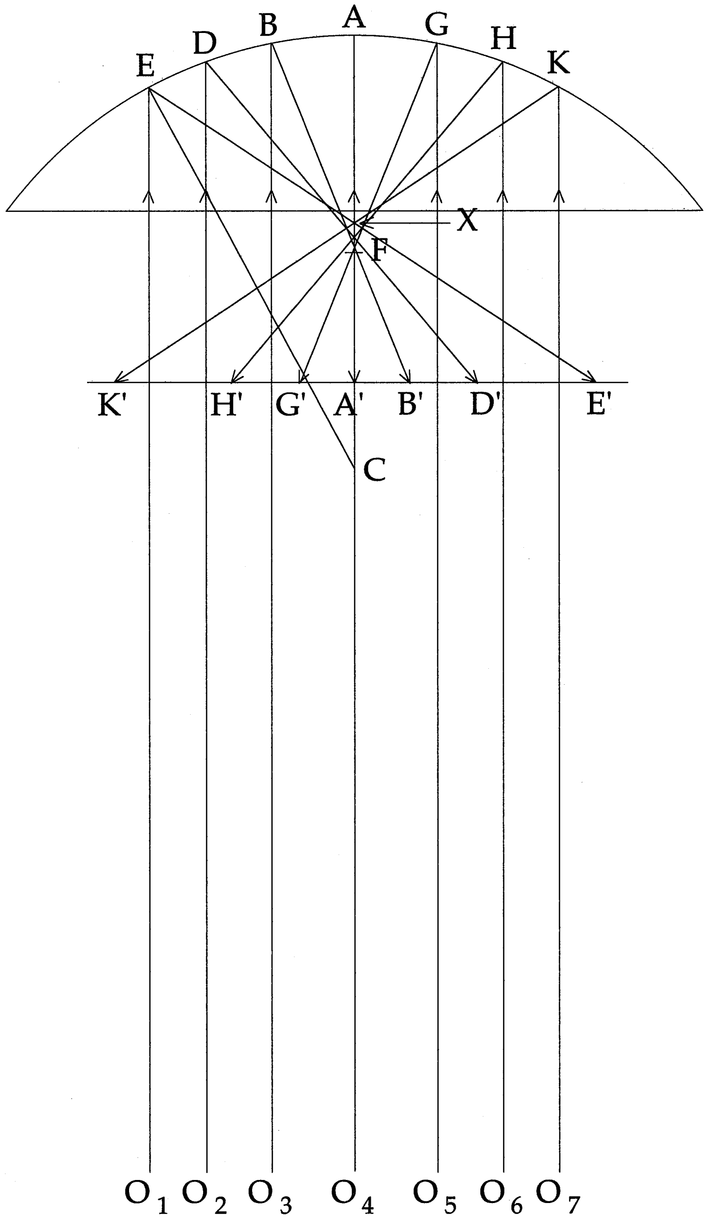

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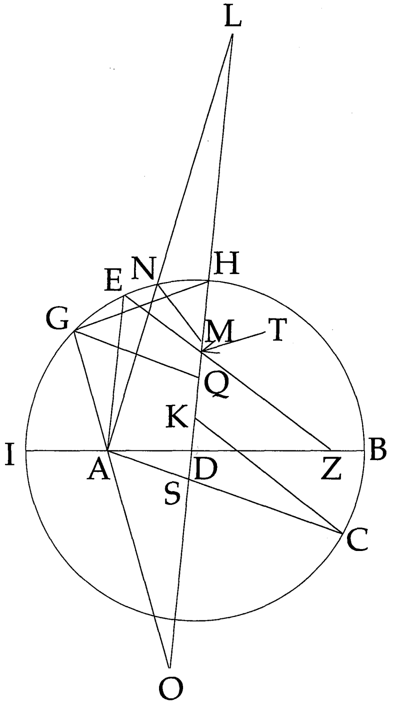

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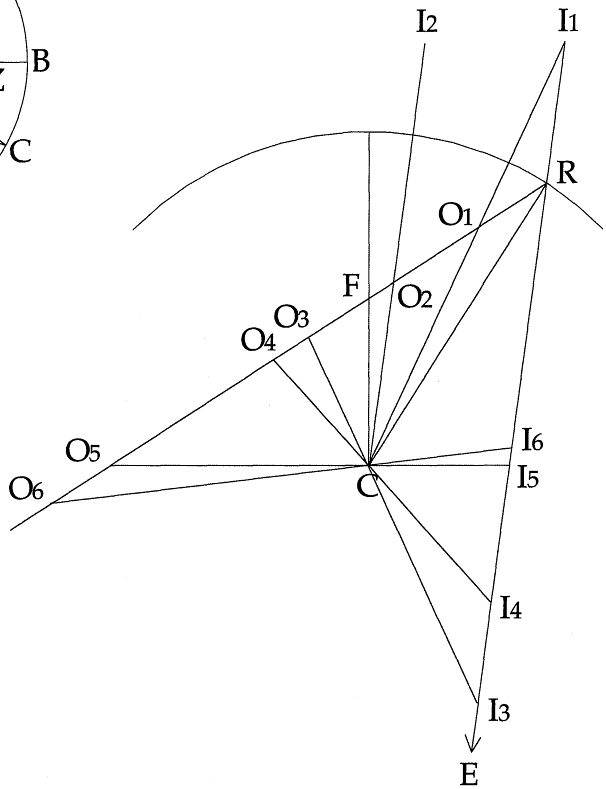

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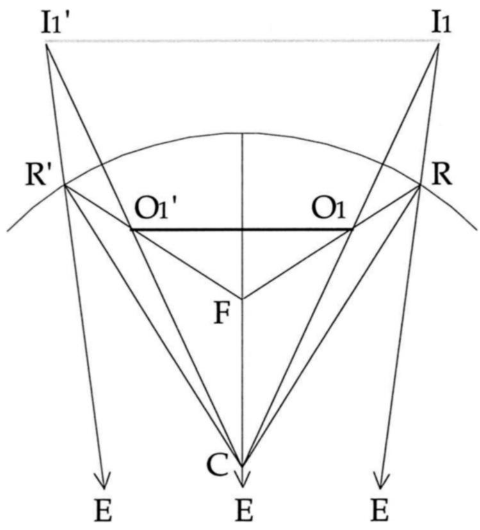

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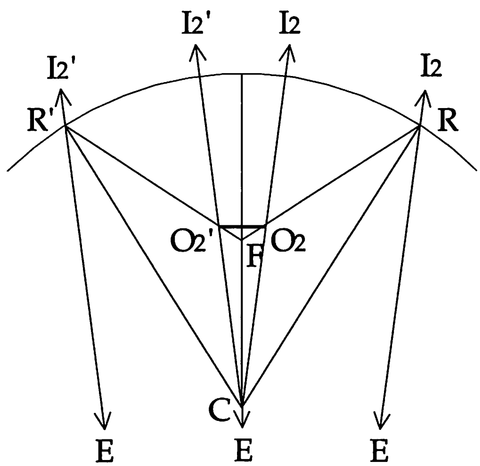

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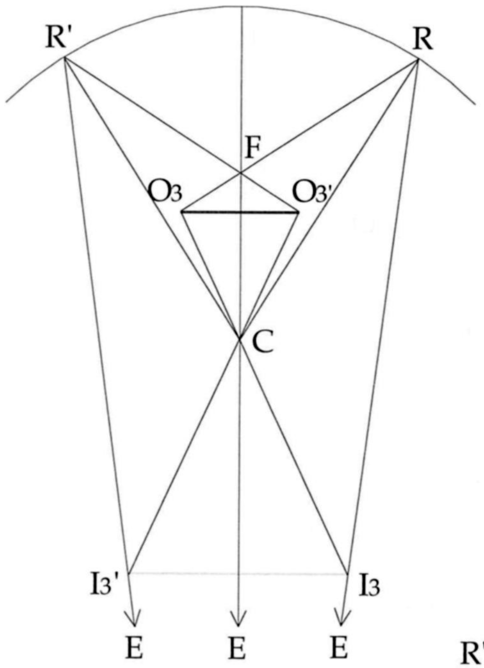

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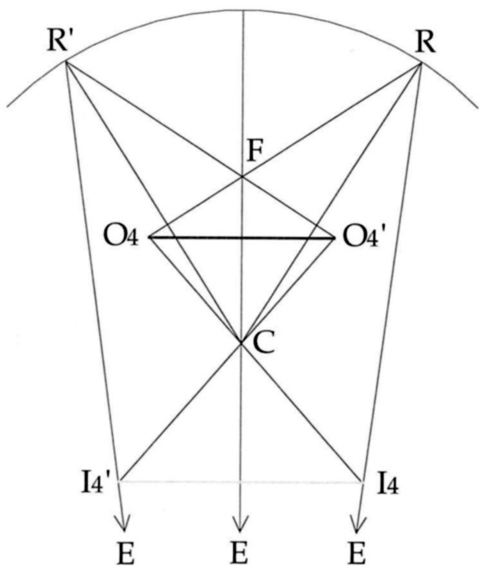

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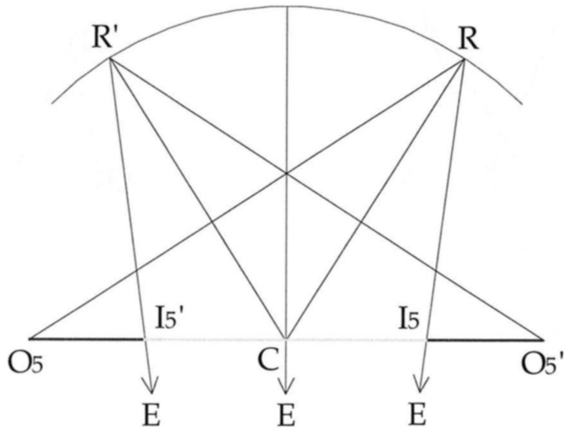

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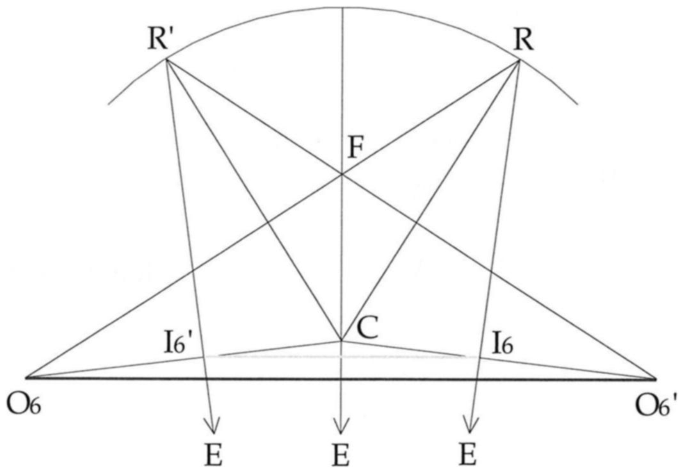

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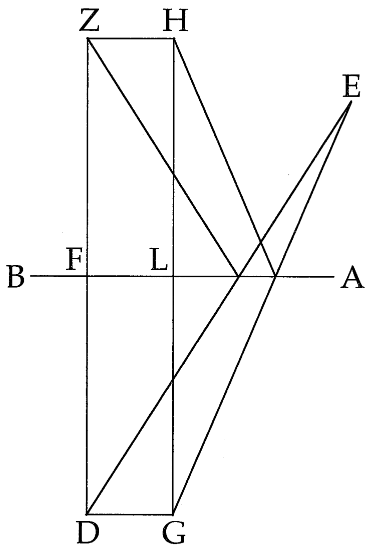


FIGURE 6.3.1

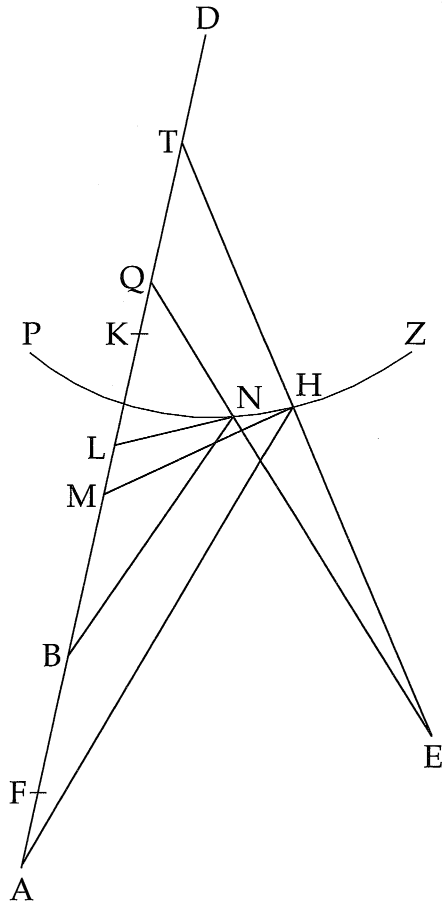


FIGURE 6.4.2

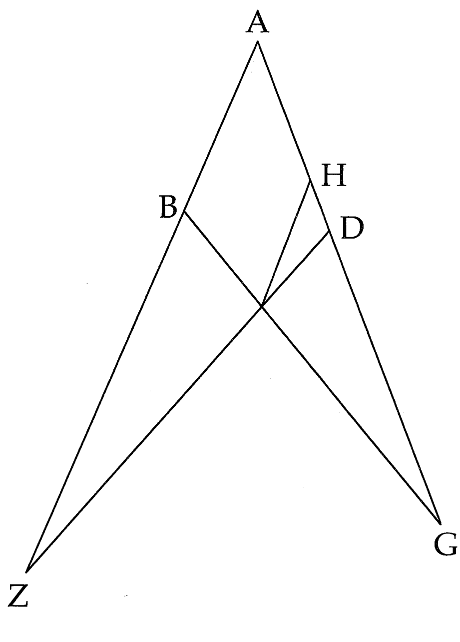


FIGURE 6.4.2a

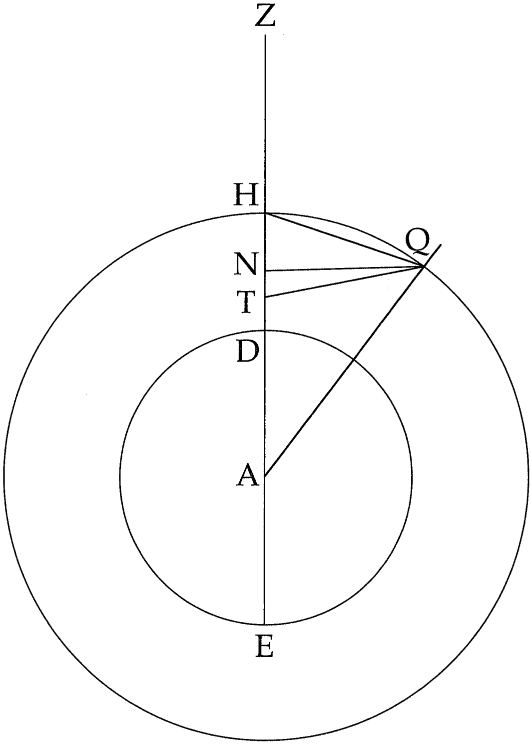


FIGURE 6.4.3

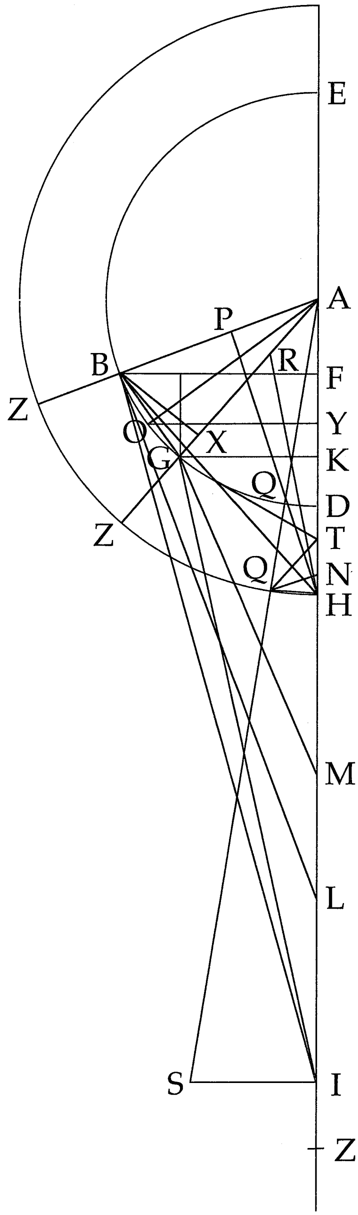


FIGURE 6.4.3a

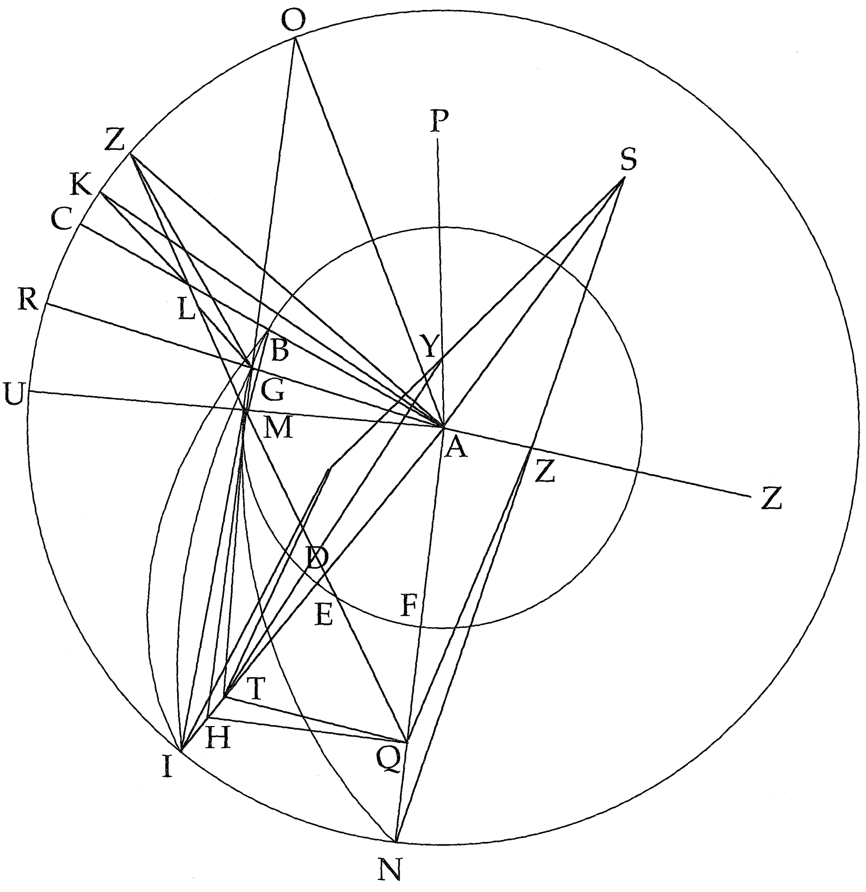


FIGURE 6.4.3b

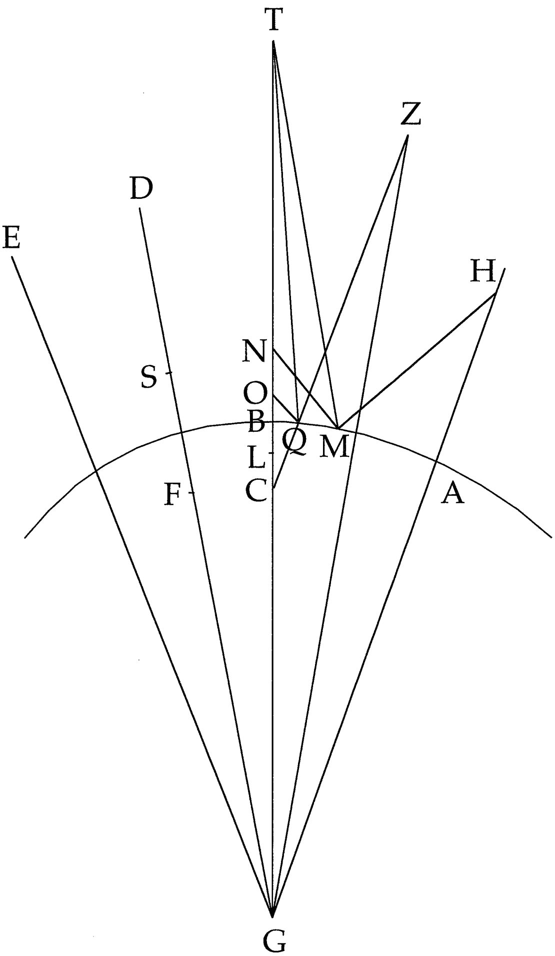


FIGURE 6.4.4

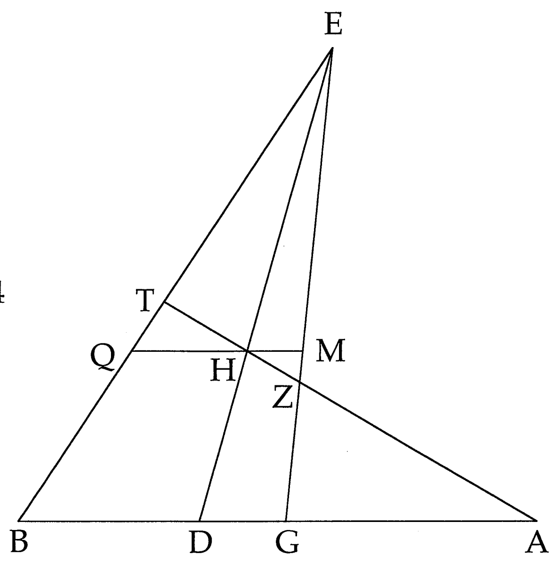


FIGURE 6.4.5

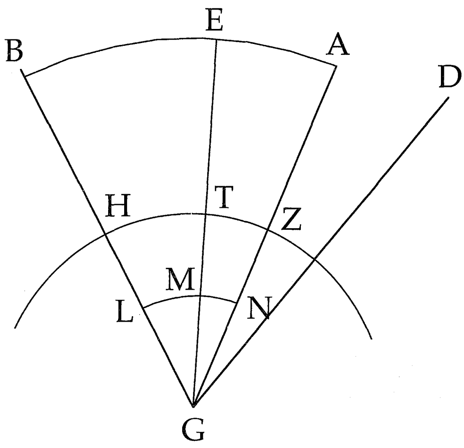


FIGURE 6.4.8

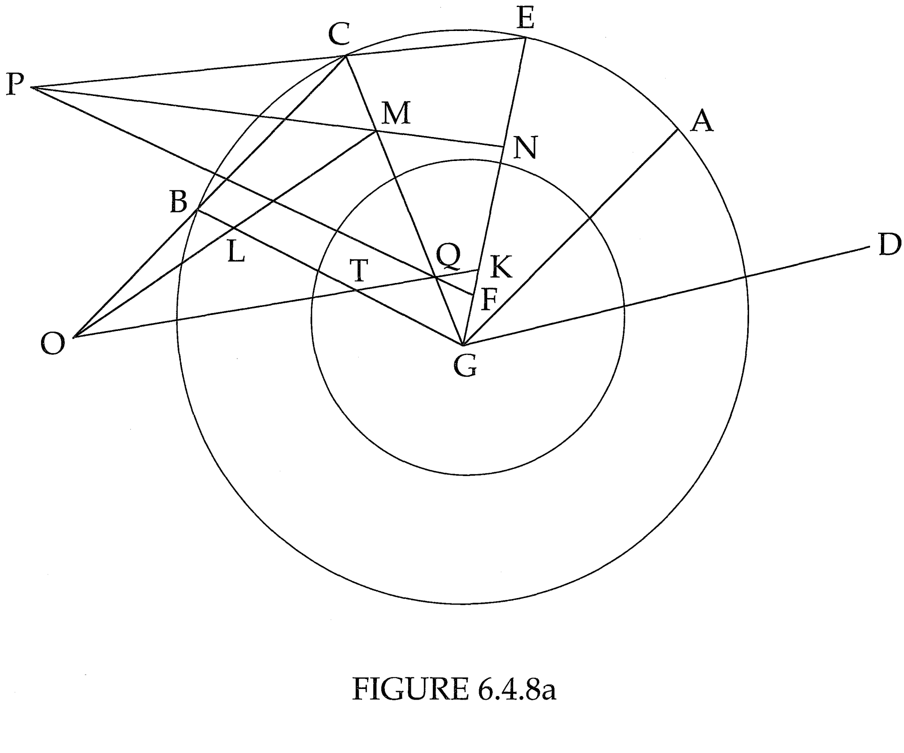


FIGURE 6.4.8a

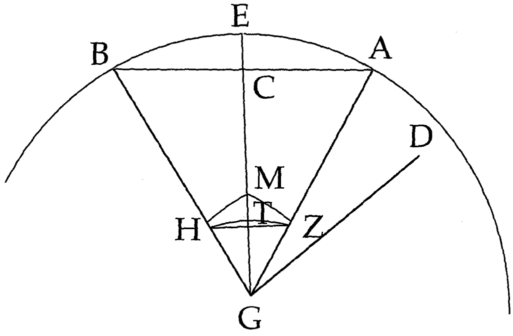


FIGURE 6.4.10

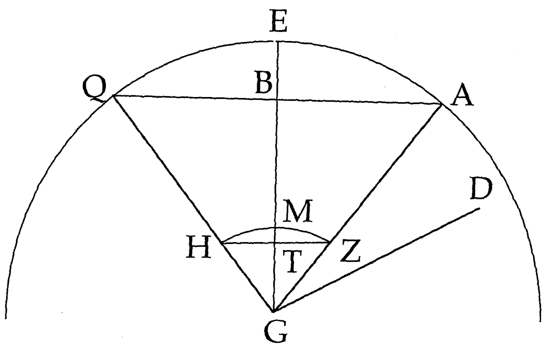


FIGURE 6.4.11

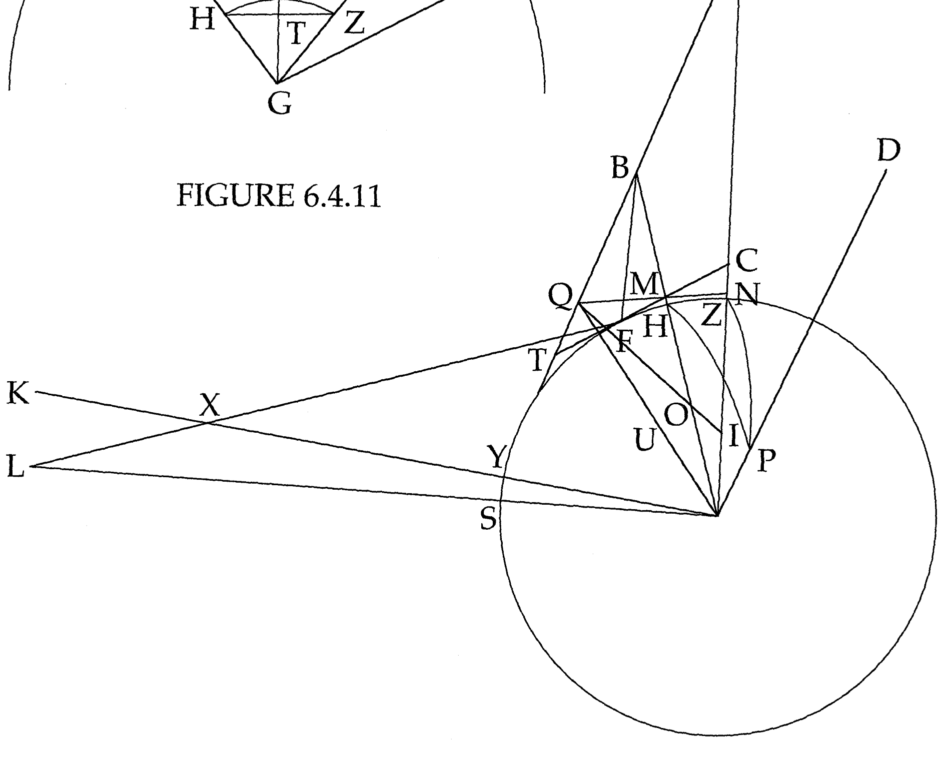


FIGURE 6.4.12

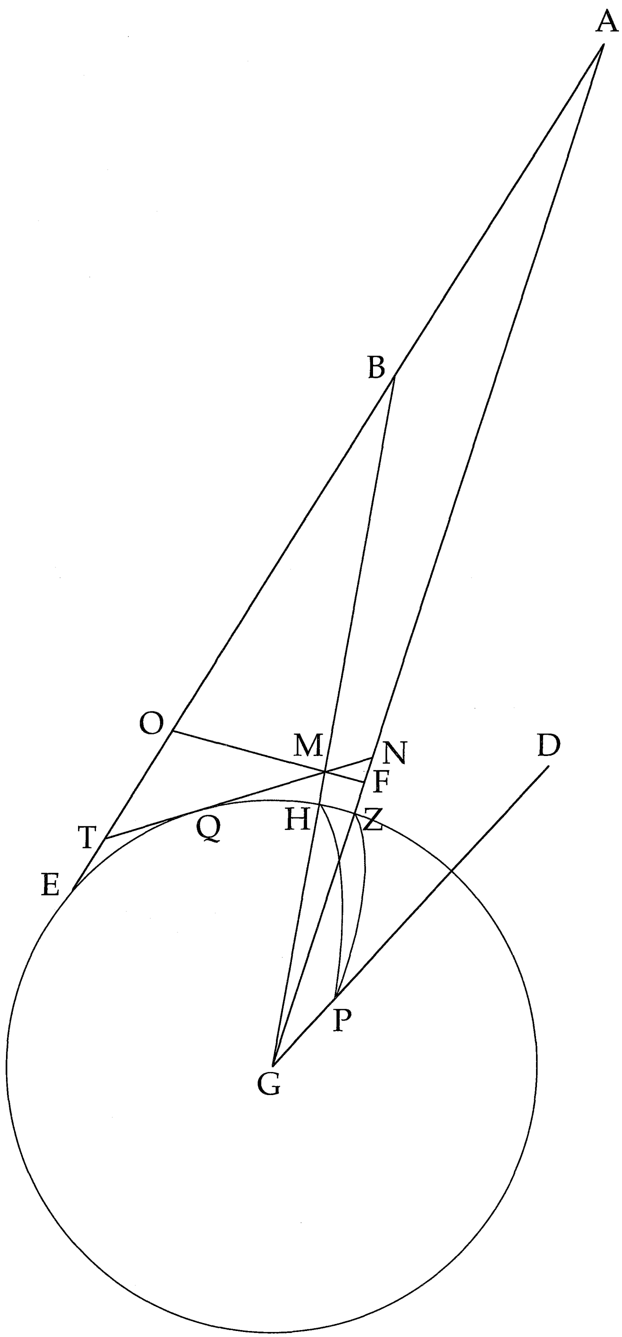


FIGURE 6.4.13

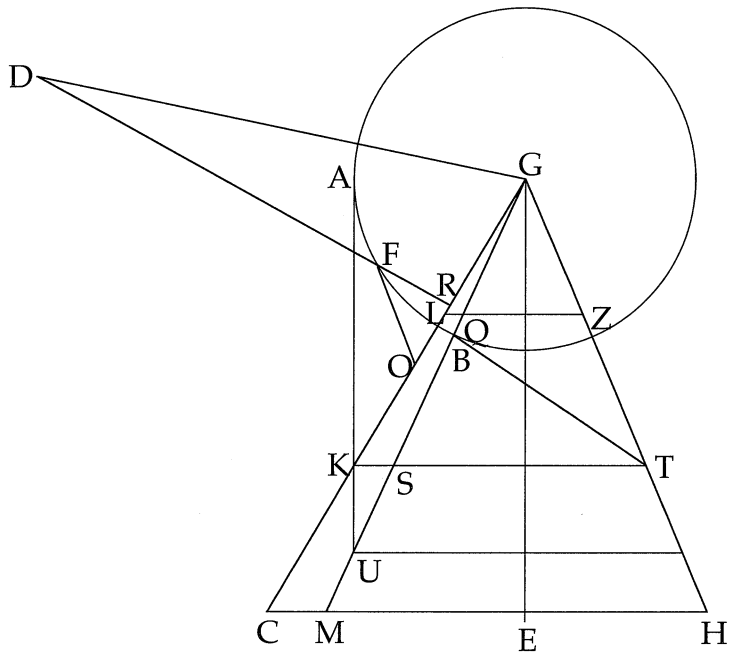


FIGURE 6.4.15

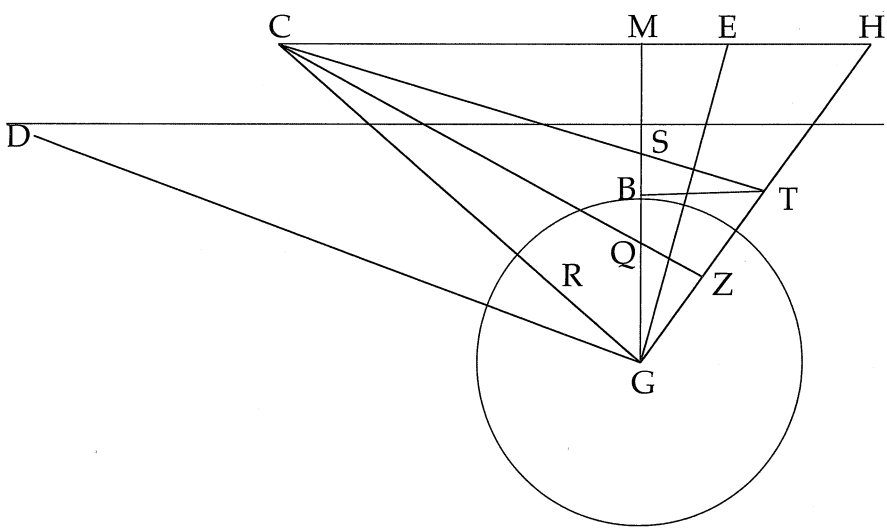


FIGURE 6.4.15a

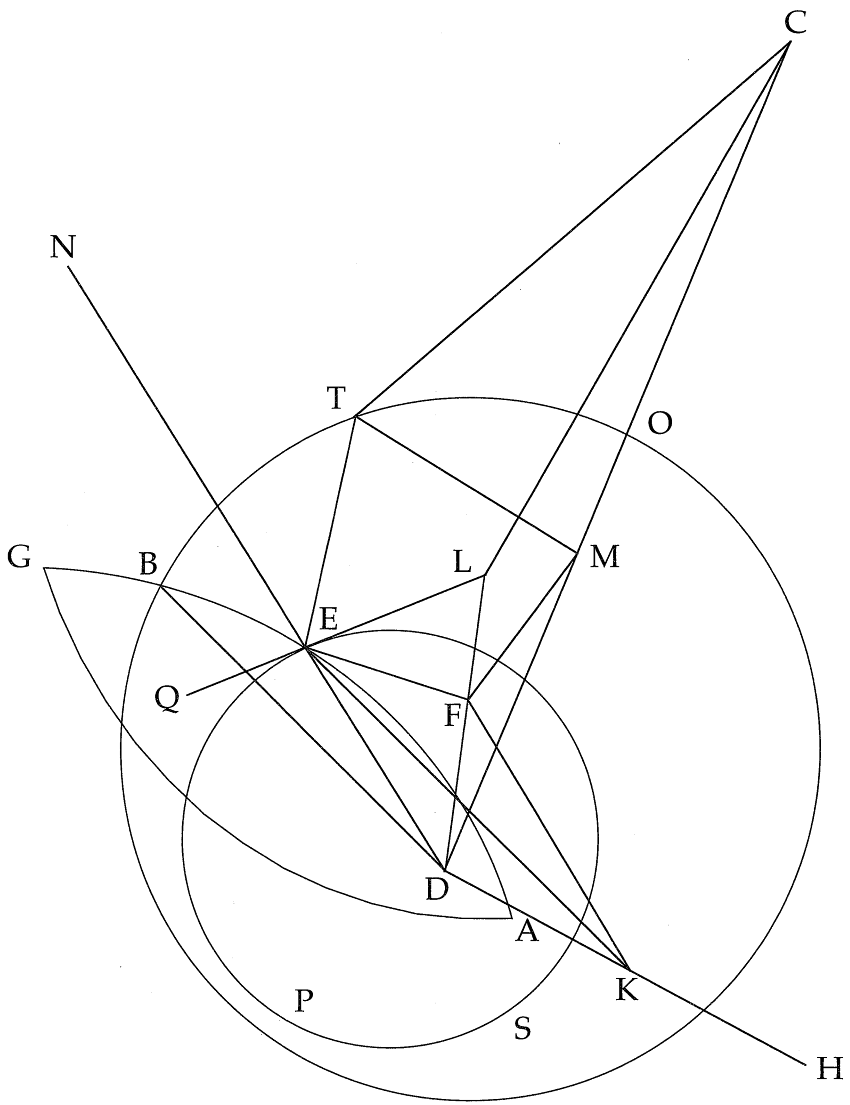


FIGURE 6.5.16

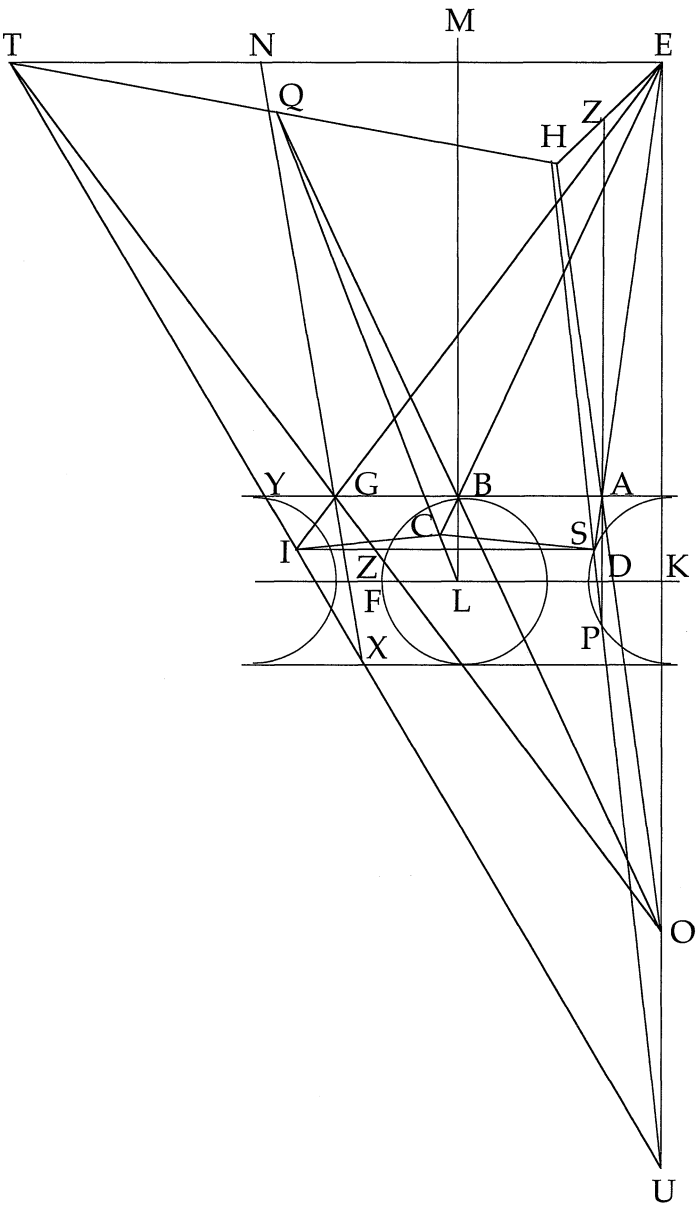


FIGURE 6.5.17

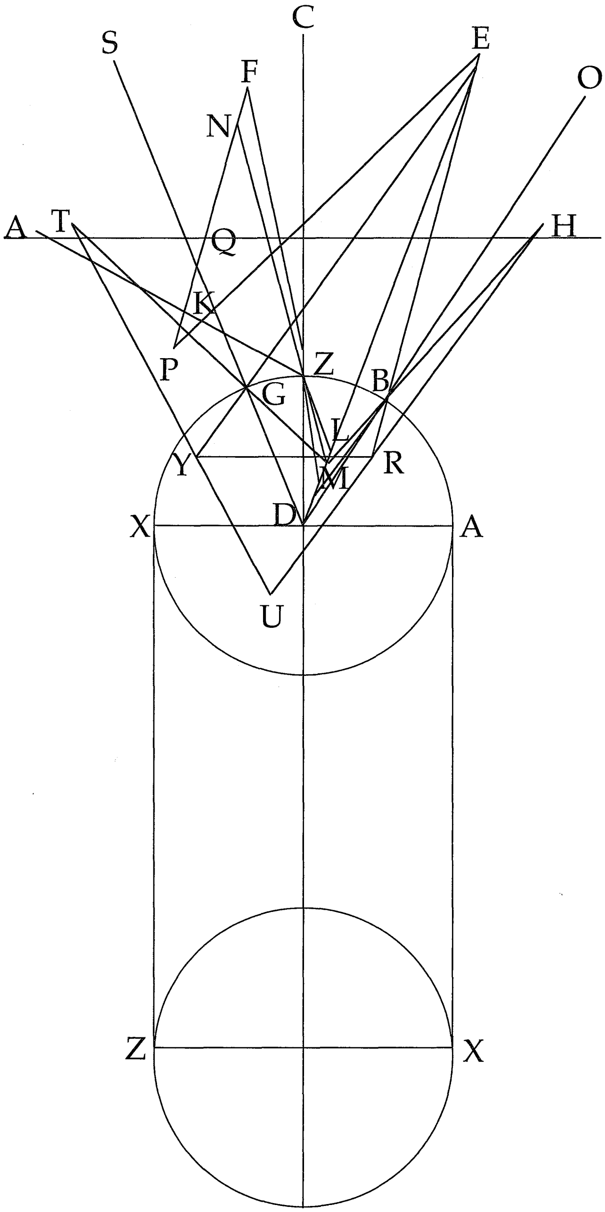


FIGURE 6.5.19

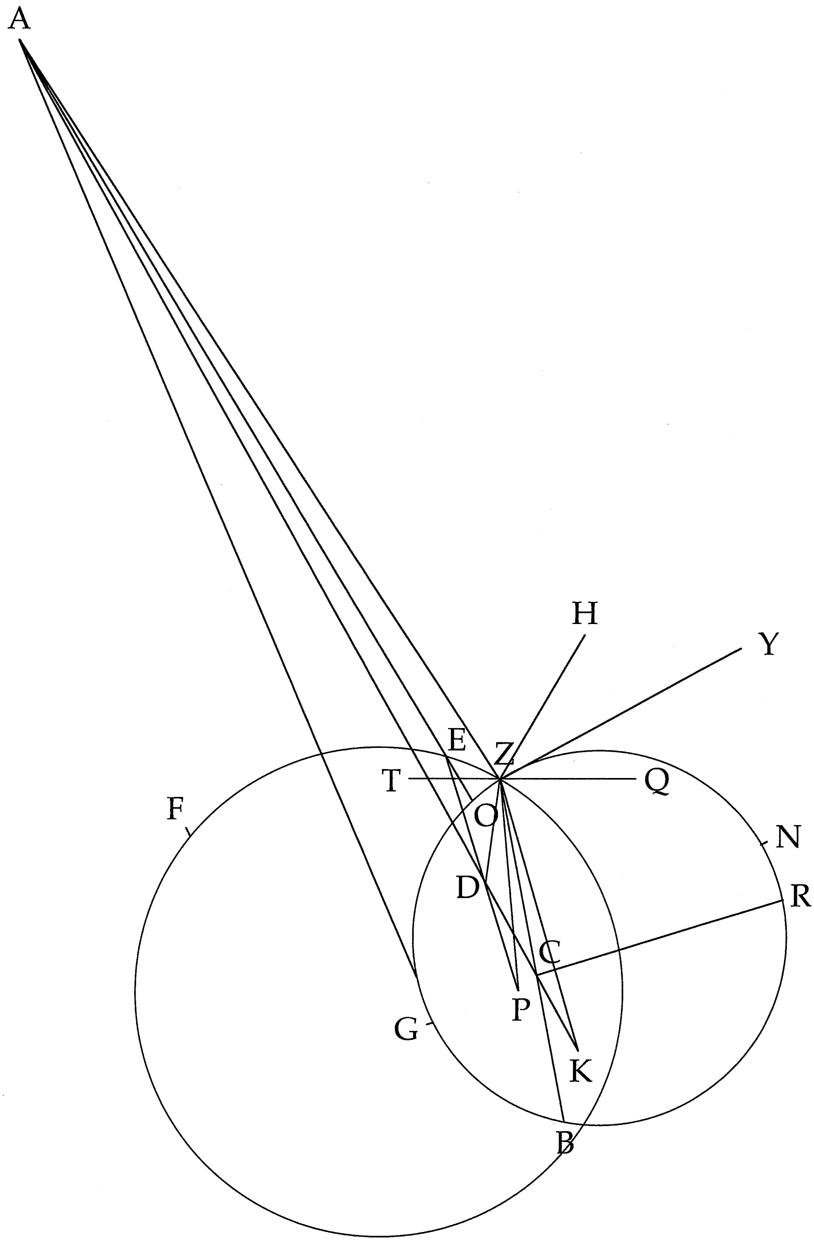


FIGURE 6.6.20

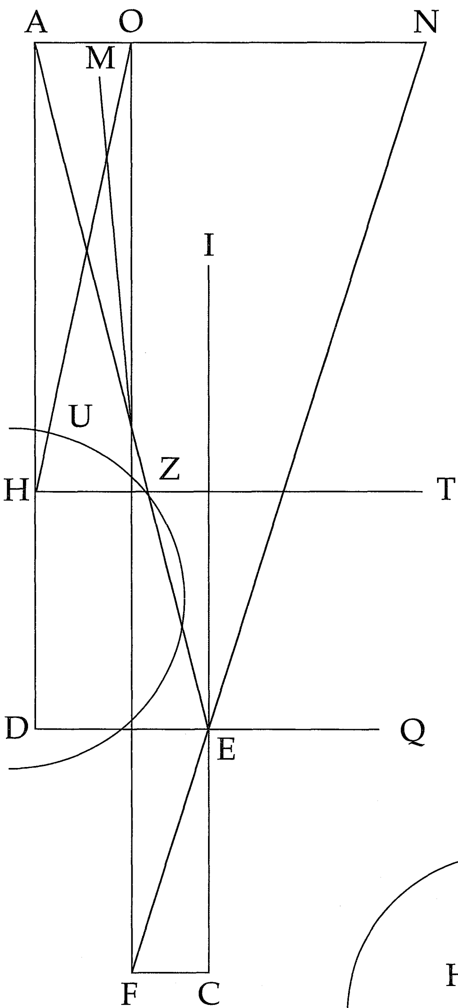


FIGURE 6.6.21

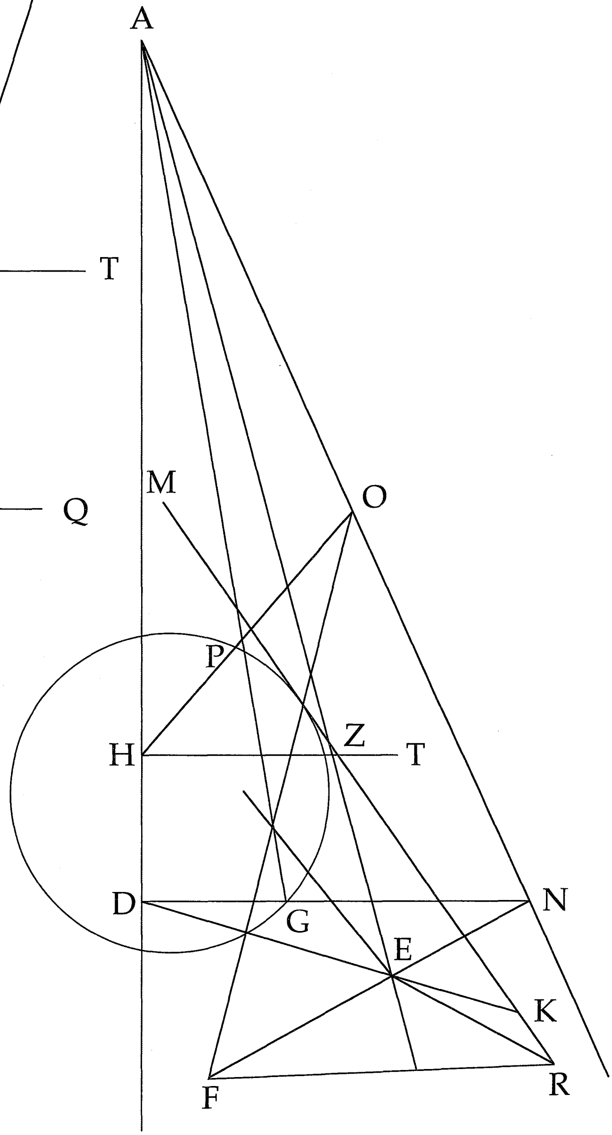


FIGURE 6.6.22

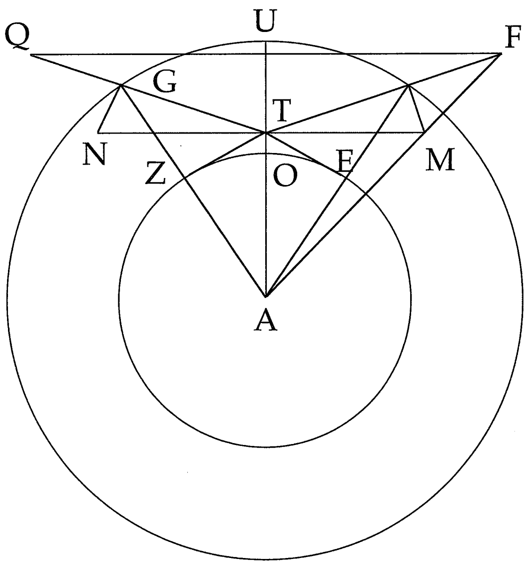


FIGURE 6.7.23

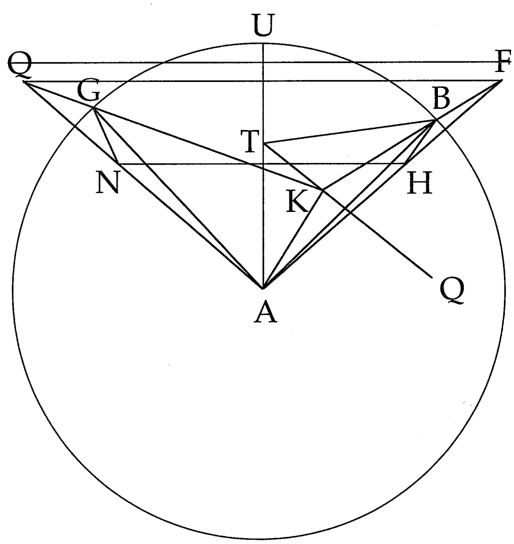
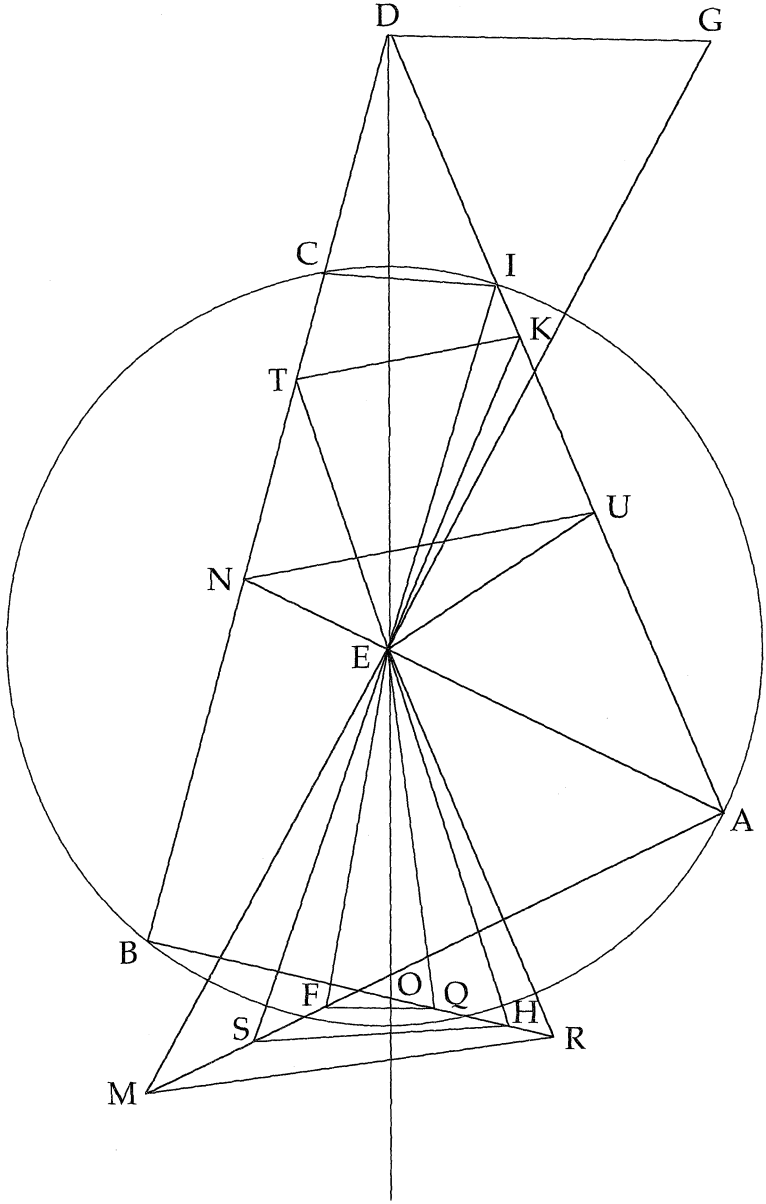


FIGURE 6.7.24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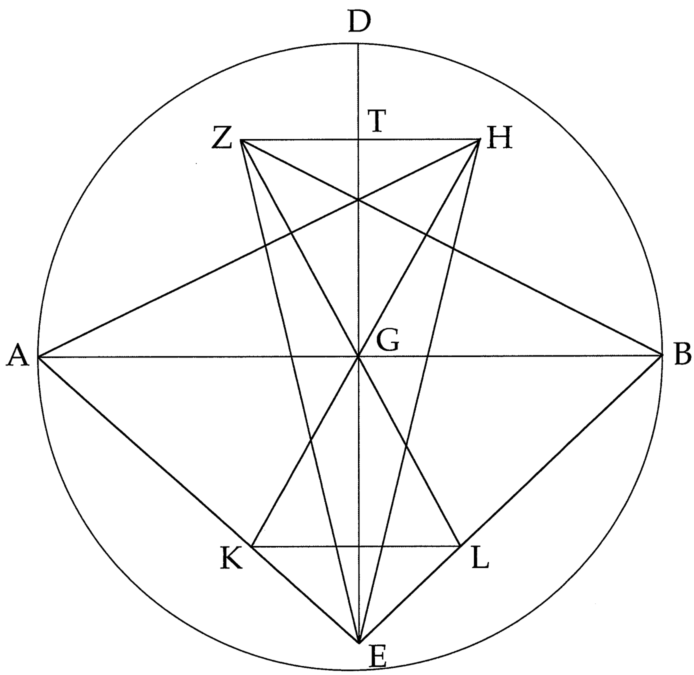

FIGURE 6.7.28

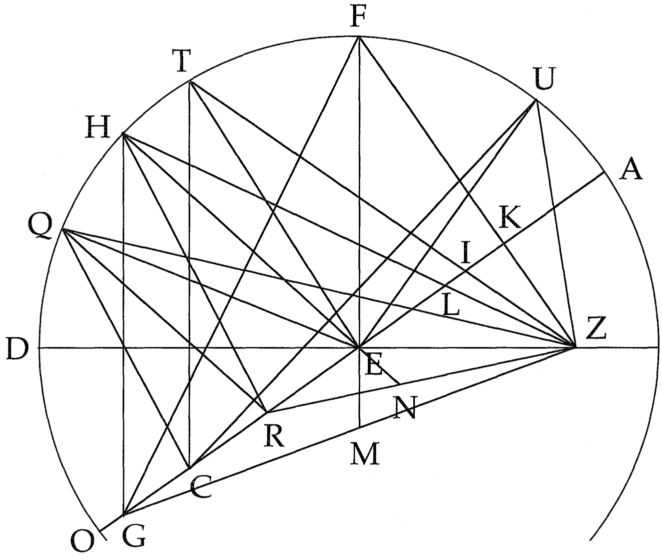

FIGURE 6.7.29

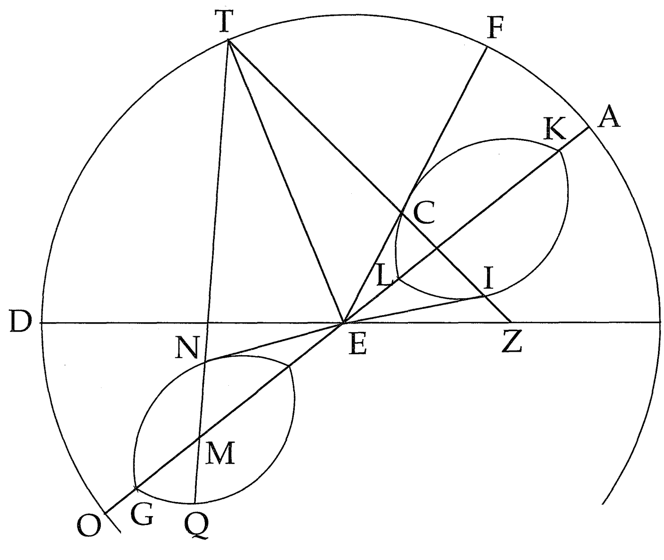

FIGURE 6.7.30

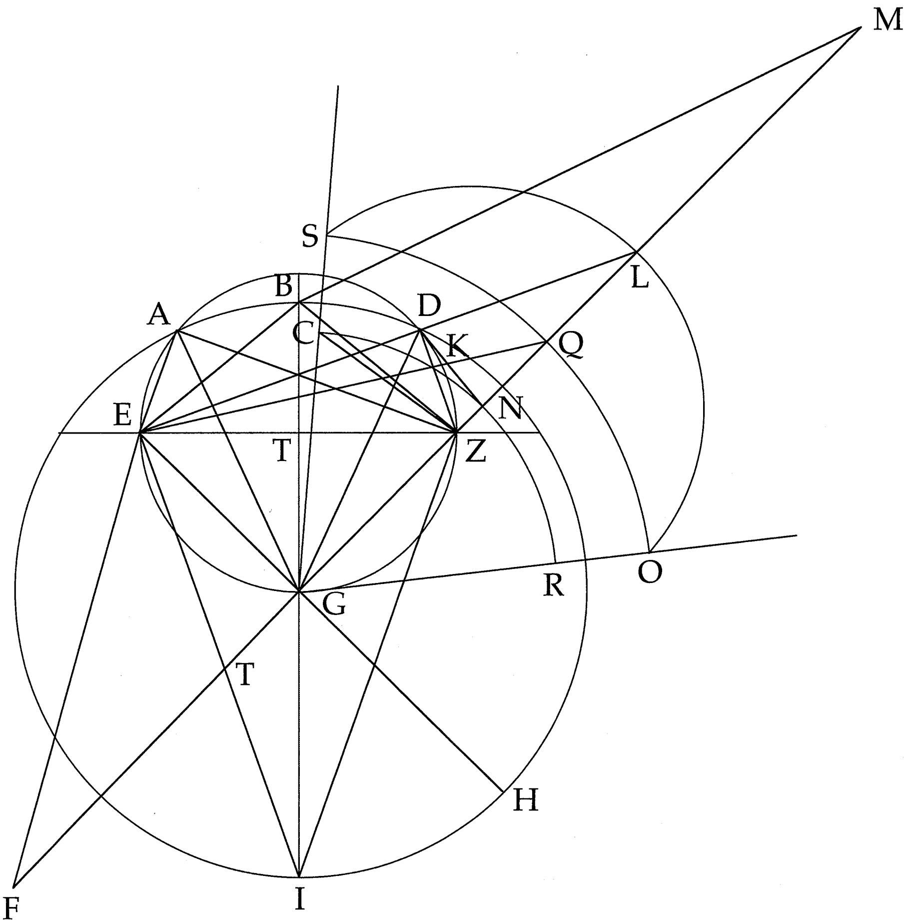

FIGURE 6.7.31

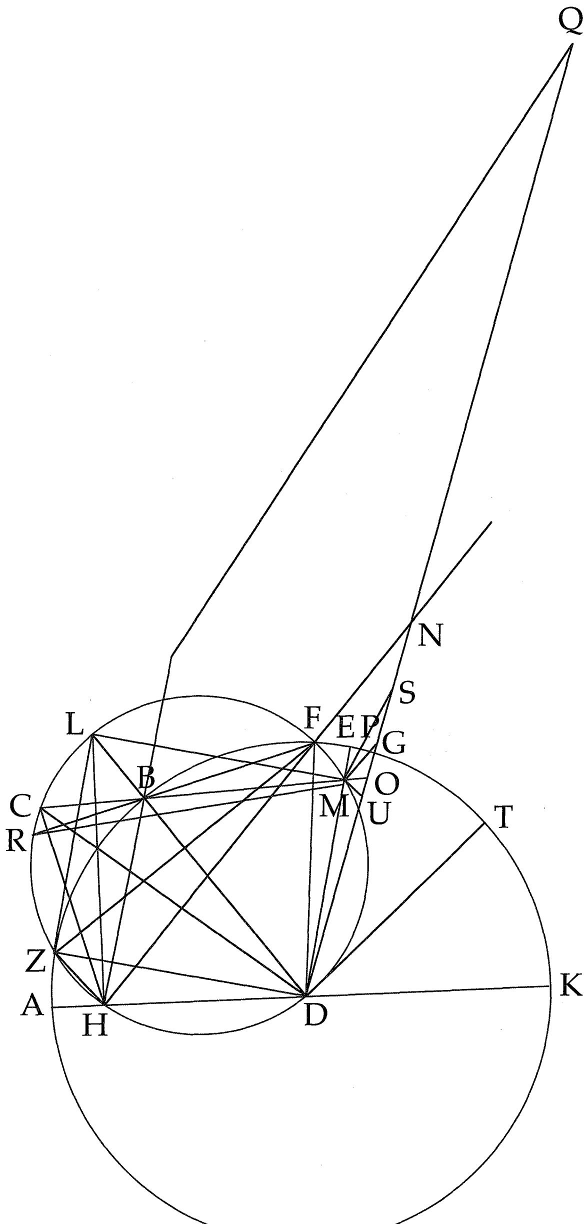

FIGURE 6.7.32

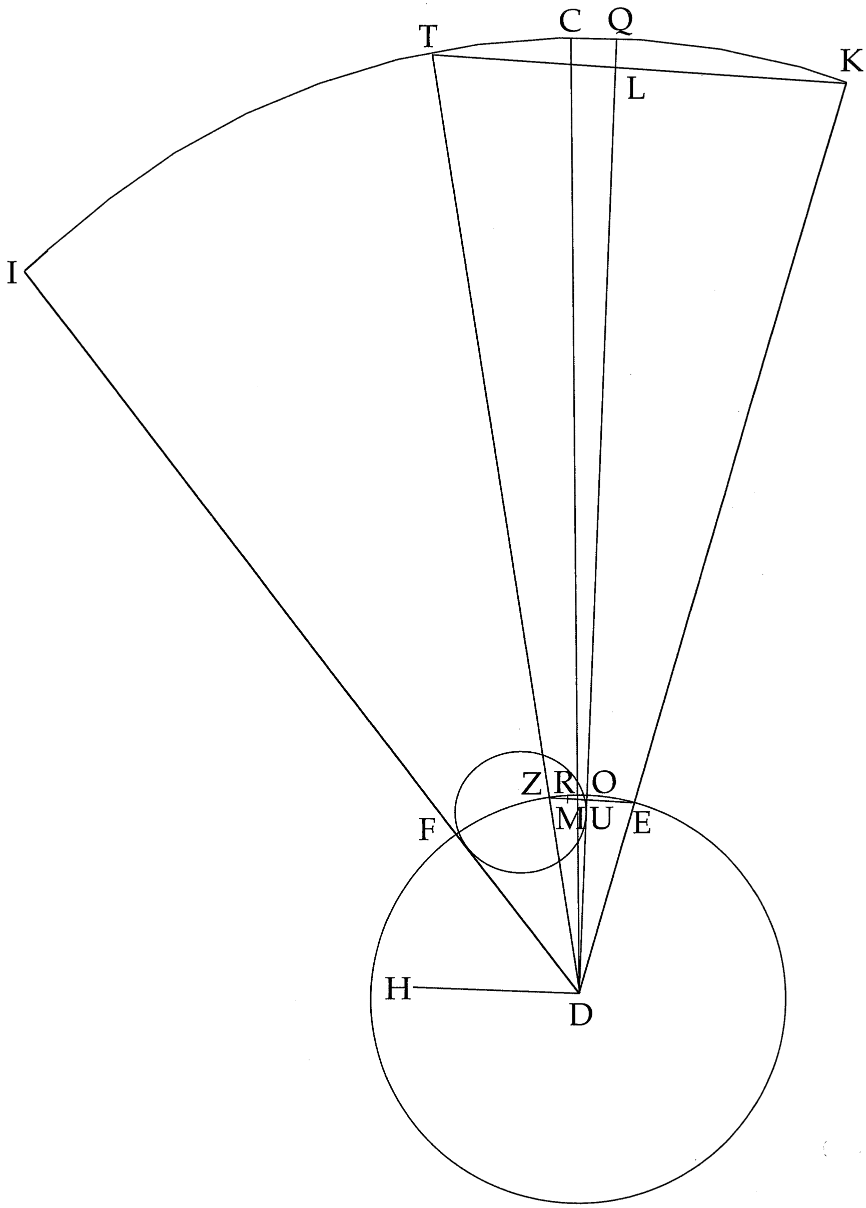

FIGURE 6.7.32a

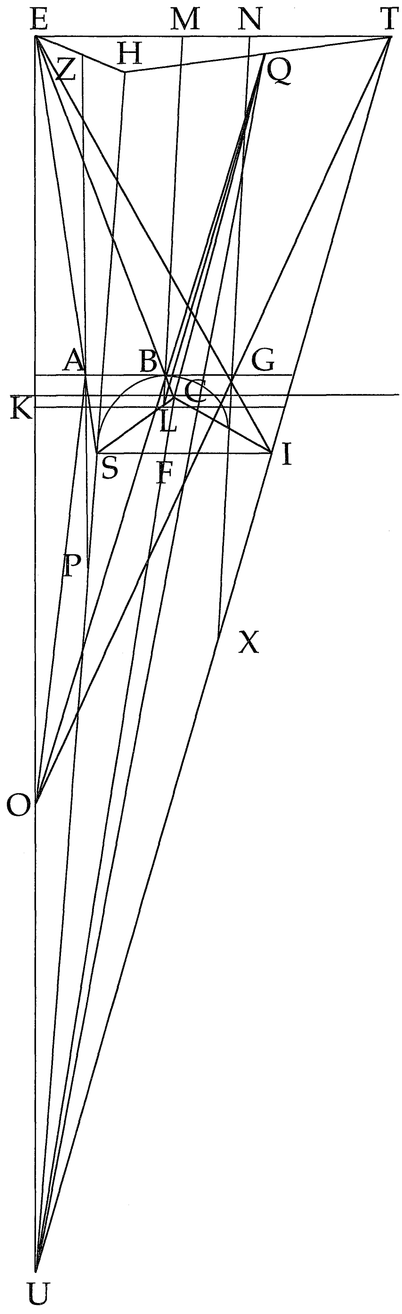

FIGURE 6.8.33

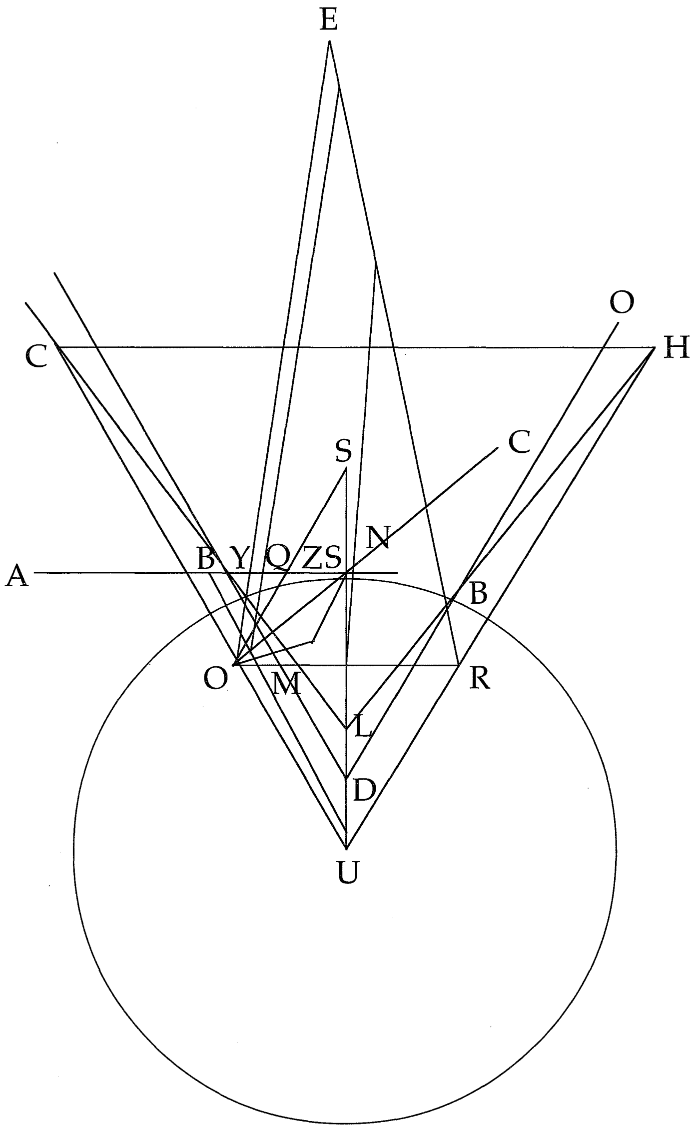

FIGURE 6.8.34

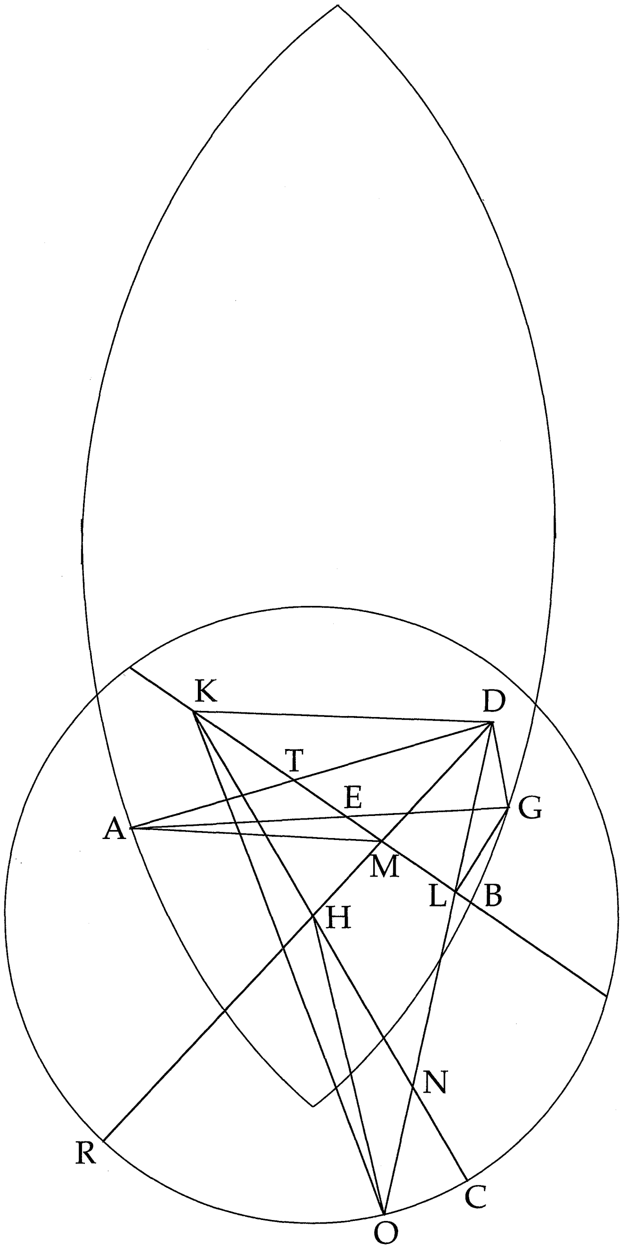

FIGURE 6.8.35

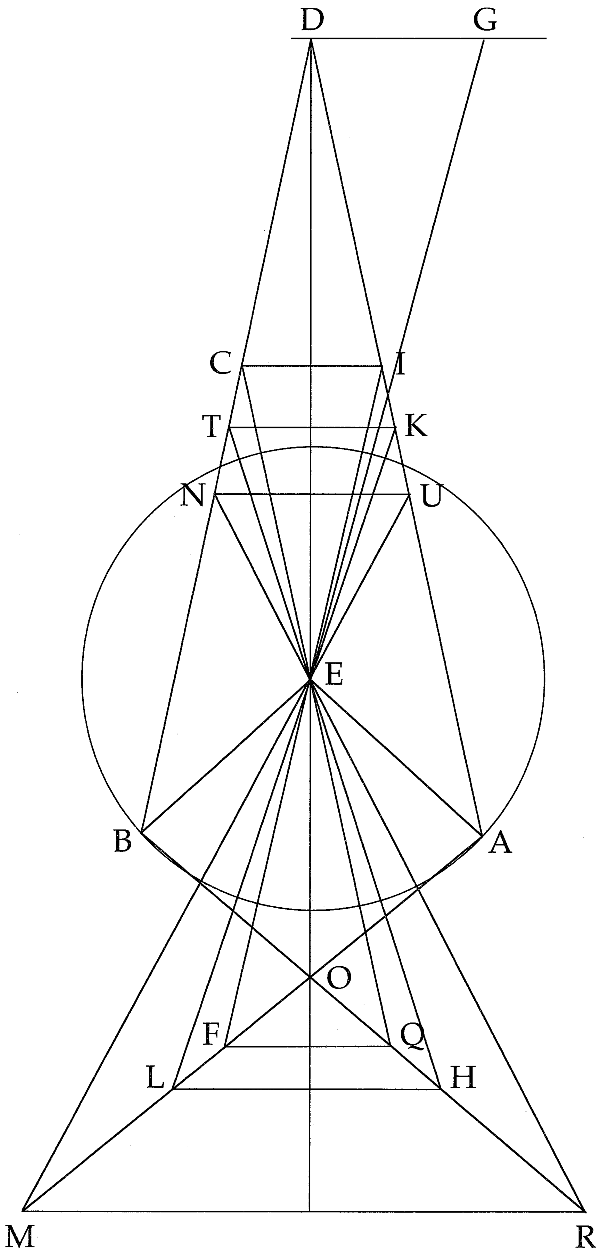

FIGURE 6.8.36

APPENDIX

CAPITULUM SEXTUM

De fallaciis que accidunt in speculis piramidalibus convexis erectis

[1] In istis autem accidunt deceptiones quarum causa est conversio, ut accidunt in speculis columpnaribus convexis, in nullo enim differunt preter quam in hoc quod forme que comprehenduntur in hiis sunt magis declinantes ad piramiditatem, et quod est in capite speculi ex forma est strictius.

[2] In ceteris vero rebus adequantur linearum rectarum, enim forme convertuntur a superficiebus istorum speculorum a lineis rectis, et ymagines earum sunt sicut ymagines linearum rectarum que equidistant vel appropinquant longitudini speculi columpnalis, scilicet quare erunt convexe parum, et convexitas earum erit ex parte visus.

[3] Et ymagines etiam linearum rectarum equidistantium latitudini speculi piramidalis erunt etiam sicut ymagines linearum rectarum que equidistant latitudini speculis columpnalis, scilicet quod linee quarum forme convertuntur erunt convexe convexitate manifesta, et erit centrum visus extra superficies in quibus est convexitas, et erunt diametri earum multum minores ipsis lineis. Ad huius autem demonstrationem premittamus hanc propositionem.

[4] **[PROPOSITIO 20]** Cum ceciderit aliquis sector in speculo piramidali convexo erecto, et in superficie eius fuerit extracta perpendicularis super superficiem contingentem piramidem in puncto conversionis, et fuerit etiam extracta alia linea perpendicularis super lineam contingentem circumferentiam sectoris in puncto remotiori a capite piramidis puncto conversionis, tunc, si ista perpendicularis extracta fuerit recte, concurret cum perpendiculari extracta a puncto conversionis super punctum existens sub puncto quod in axe.

[5] Sit ergo speculum ABG [figure 6.6.20alt, p. 153], et caput eius A, et cadat in ipsum sector ex sectoribus a quorum circumferentia

4 istis: quibus *O* / conversio: convexio *L3* 7 ex forma est: est ex forma *L3* 9 rectarum enim *transp.* (rectarum *inter.*) *O* 10 speculorum: speculum *L3* 11 earum: eorum *L3* 12 vel appropinquant *inter. O* / columpnalis: columpnaris *L3* 18 erunt: erit *L3* 19 *post* visus *add.* et *L3* 20 multum: multo *L3* 21 propositionem: proportionem *OL3* 22 sector: sectior *L3* 23 erecto: erecta *L3* / in *inter. O* 26 sectoris: sectionis *L3* 27 piramidis: piramidali *L3* 28 extracta[1]: exstracta *O* 29 *post* quod *add.* est *L3* 31 ipsum: speculum *L3* / sector: sectoris *O*

CHAPTER SIX

Concerning the misperceptions that occur in right convex conical mirrors

[1] In these [sorts of mirrors] there occur the misperceptions whose cause is reflection [itself], as happens in convex cylindrical mirrors, for [those mirrors] differ only insofar as the forms that are perceived in them are more inclined toward a conical shape, and the part of the form that lies at the vertex of the mirror is most acutely narrowed.

[2] As for the remaining phenomena, however, they are appropriately explained by rectilinear radiation, for forms are reflected from the surfaces of these mirrors according to straight lines, and their images are like the images of straight lines that are parallel or nearly parallel to the [length along the] longitude of a cylindrical mirror, i.e., because they will be somewhat convex, and their convexity will lie toward the center of sight.

[3] In addition, the images of straight lines posed along the width of conical mirrors will also be like the images of straight lines posed along the width of cylindrical mirrors, that is, the lines whose forms are reflected will be convex with a pronounced curvature, and the center of sight will lie outside the surface containing the convexity, and their cross-sections will be much shorter than the lines themselves. To demonstrate this, let us set forth this proposition.

[4] **[PROPOSITION 20]** When a given [conic] section falls on [the surface of] a right convex conical mirror, and when a normal is dropped to the plane tangent to the cone's surface at the point of reflection, and when another line is also dropped perpendicular to a line tangent to the periphery of the [conic] section at a point farther from the cone's vertex than the reflection-point, then, if this latter perpendicular is extended in a straight line, it will intersect the normal dropped from the point of reflection at a point lying outside [any] point on the [cone's] axis.

[5] Accordingly, let ABG [in figure 6.6.20alt] be the mirror, A its vertex, and let one of the conic sections from whose periphery forms are reflected

convertuntur forme. Sit ergo BFZ, et punctum conversionis sit E, et sit perpendicularis exiens a puncto conversionis in superficie sectoris linea ED. Et tangat linea TZQ sectorem BFZ in Z, et sit Z remotius a puncto A quam E. Et transeat per Z superficies equidistans basi piramidis, et faciet circulum LZB. Iste ergo circulus secat sectorem BFZ, quia circulus est perpendicularis super axem AD, et sector est obliquus super ipsum.

[6] Et continuemus ZA, AE, et extrahamus AE donec concurrat cum circumferentia LZB in O. O ergo est remotius a puncto A quam E, quia AO est sicut AZ. Cum ergo exiverit ex O perpendicularis super superficiem contingentem superficiem piramidis transeuntis lineam AO, concurret cum axe ultra punctum D. Sit ergo concursus K, et continuemus KZ. Erit ergo perpendicularis super superficiem contingentem piramidem transeuntem per AZ, quia angulus AZK est rectus, et linea QZ est in hac superficie contingente. Angulus ergo QZK est rectus.

[7] Et extrahamus ex Z differentiam communem superficiei circuli et superficiei contingenti piramidem, et sit ZY. Et sit centrum circuli C, et continuemus CZ. Angulus ergo CZY est rectus. Et extrahamus ex C lineam in superficie circuli continentem cum linea CZ angulum rectum, et sit CL. Linea ergo CL est equidistans linee ZY, et linea CL est perpendicularis super superficiem AZK, quia angulus ZCL est rectus positione. Et angulus ZCA est rectus, quia AC est perpendicularis super superficiem circuli. Linea ergo ZC est perpendicularis super superficiem ACL. Et ZY est equidistans CL; linea ergo ZY est perpendicularis super superficiem AZC. Ergo linea ZQ est obliqua super superficiem AZC.

[8] Et extrahamus a puncto Z in superficie sectoris lineam continentem cum linea ZQ angulum rectum, et sit ZH. Et quia D in superficie sectoris ex axe est aliud a puncto K, ut predictum est, erit K extra superficiem sectoris. Sed DZ est in superficie sectoris; ergo KZ est

32 ergo *om. L3* / BFZ: BEG *O*; FZ *L3* 34 TZQ: TZK *O*; CZQ *L3* / BFZ: BEF *L3*; *alter. ex* BG *in* BEG *O* 35 per: super *L3* 36 *post* LZB *inter.* F *O*; *add.* FG *L3* 37 BFZ: BEG *O*; BEF *L3* 38 obliquus: oblitus *L3* 39 *post* continuemus *add.* DZ *O* / ZA: AZ *O*; DZA *L3* / AE[1]: ZAE *L3* / AE[2]: AZ *L3* 40 LZB: LZF *O*; LZG *L3* / O[2] *om. O* / O ergo *transp. L3* 41 exiverit: exierit *L3* 43 concurret: concurrat *L3* / sit ergo *transp. O* 44 K: Q *O* / et *om. L3* / KZ: QZ *deinde inter.* F in arabico *O* / *post* KZ *add.* DZ *L3* / ergo *inter. O* 45 AZK: AZQ *OL3* 46 QZ: KZ *OL3* / contingente: contingentem *O* 49 ZY: ZK *O*; *alter. ex* ZRQY *in* ZRY *L3* / et[3] *om. L3* / et sit[2] *inter. O* 51 ex: in *L3* / circuli *om. L3* / continentem: contingentem *L3* 52 CL[1]: OZ *O*; CZ *L3* / CL[2]: CZ *OL3* / ZY: ZK *OL3* 53 CL: CZ *OL3* / AZK: AZC *OL3* / ZCL: AZ *L3* 56 ACL: AZQ *OL3* / ZY[1,2]: ZF *L3* / est equidistans *mg. O* / CL: CZ *OL3* (*mg. O*) 57 AZC: AZQ *OL3* 58 AZC: AZQ *OL3* 59 Z: DZ *OL3* / continentem: contingentem *L3* 60 ZQ: ZK *OL3* 61 est[1] *om. L3* / aliud: illud *L3* / K[1]: E *OL3* / K[2]: Q *O*; *om. L3* 62 DZ *corr. ex* DEZ *L3* / KZ: QZ *O*; quia *L3*

fall on it. Let it be BFZ, let E be the point of reflection, and let the normal dropped from the point of reflection within the plane of the [conic] section be line ED. Let line TZQ be tangent to [conic] section BFZ at Z, and let Z lie farther from point A than E [does]. Then let a plane pass through Z parallel to the base of the cone, and it will form circle LZB. This circle therefore intersects [conic] section BFZ, because the circle is perpendicular to [the cone's] axis AD, and the [conic] section is oblique to it.

[6] Let us then draw ZA and AE, and let us extend AE until it intersects the circumference [of circle] LZB at O. O thus lies farther from point A than E [does], because AO = AZ. Hence, if a normal is dropped from O to the plane tangent to the surface of the cone and passing along line AO, it will intersect the axis below point D. Let K be the [point] of intersection, then, and let us draw KZ. It will therefore be perpendicular to the plane tangent to the cone and passing through AZ, because angle AZK is right, and line QZ lies in this tangent plane. Angle QZK is therefore right.

[7] Now from Z let us draw the common section of the circle's plane and the plane tangent to the cone [at Z] and let it be ZY. Let C be the center of the circle, and let us draw CZ. Angle CZY is therefore right. Let us then draw a line from C in the plane of the circle that forms a right angle with line CZ, and let it be CL. Line CL is thus parallel to line ZY, and line CL is perpendicular to plane AZK, because angle ZCL is right, by construction. Angle ZCA is also right, because AC is perpendicular to the plane of the circle. So line ZC is perpendicular to plane ACL. In addition, ZY is parallel to CL, so line ZY is perpendicular to plane AZC. Consequently, line ZQ is oblique to plane AZC.

[8] From point Z let us now draw a line within the plane of the [conic] section that forms a right angle with line ZQ, and let it be ZH. Since D on the axis within the plane of the [conic] section is distinct from point K, as was claimed earlier, K will lie outside the plane of the [conic] section. But DZ lies in the plane of the [conic] section, so KZ lies outside the plane of the [conic] section. ZH, on the other hand, lies in the plane of

extra superficiem sectoris. Sed ZH est in superficie sectoris; ergo KZ secat ZH, nec continuatur cum ipsa. Sunt ergo in eadem superficie preter superficiem sectoris secante superficiem sectoris super lineam ZH. Ergo ZD, que est in superficie sectoris, est extra superficiem in qua sunt linee KZ, ZH.

[9] Sed ZK continet cum ZQ angulum rectum, quia ZK est perpendicularis super superficiem contingentem piramidem, que transit per lineas AZ, ZQ. Sed linea QZ continet cum ZH angulum rectum. Ergo ZQ est perpendicularis super superficiem in qua sunt linee HZ, ZK. Et superficies QZH secat superficiem AZQ super lineam ZQ, et secat lineam KZ in Z.

[10] Et puncta T, Z, K sunt a lateribus superficiei HZQ. Ergo superficies HZQ secat superficiem in qua sunt linee DZ, ZK. Differentia ergo communis superficiebus DZK, HZQ est in superficie HZQ. Continet ergo cum ZQ angulum rectum, nam ZQ est perpendicularis super superficiem HZK, et differentia communis hiis superficiebus est media inter duas lineas ZK, ZD. Ergo angulus HZD est obtusus, et linea HZ est in superficie in qua sunt linee DZ, ZQ, que est superficies sectoris, et continet cum ZQ angulum rectum.

[11] Ergo ZH extracta recte in parte Z secabit angulum DZK, et linea HZ concurret cum ED, nam declaratum est in quinto tractatu, in undevicesima figura, in capitulo de ymagine, quod omnes due perpendiculares super duas lineas contingentes sectorem debent concurrere ultra arcum sectoris in qua est punctum contactus. Cum ergo linea HZ concurrit cum ED, et HZ secat angulum DZK, ergo linea HZ concurret cum ED sub puncto D, et hoc est quod voluimus.

63 ZH: HZ *L3* / KZ: QZ *OL3* 65 preter *alter. in* post *O* / secante: secantem *L3* 66 *post* ZH *inter.* G in arabico *O; add.* verbi gratia HZG secantem *L3* 67 KZ: QZ *O;* quia *L3* 68 ZK: ZQ *OL3* 70 ZQ: ZK *OL3* / QZ: KZ *OL3* 71 ZQ: ZK *OL3* 72 ZK: ZQ *OL3* 73 *post* KZ *add.* CQ *O* 74 et *om. L3* / HZQ: HZIQ *OL3* 77 ZQ[1,2]: ZK *OL3* 78 HZK: HZQ *OL3* / *post* hiis *add.* duabus *L3* 79 est[1]: cum *L3* / ZK: KZ *L3* / HZD: KZD *OL3* 81 ZQ: ZK *OL3* 82 ZH: ZKH *OL3* / recte: recto *O* / et *inter. O* 83 ED *corr. ex* D *O* 85 debent: dabent *O* 87 concurrit: concurret *O* 88 concurret: concurrit *L3* / est *om. L3*

the [conic] section, so KZ intersects ZH and is not continuous with it. They therefore lie in the same plane outside the plane of the [conic] section and intersecting the plane of the [conic] section along line ZH. ZD, which lies in the plane of the [conic] section, therefore lies outside the plane containing lines KZ and ZH.

[9] But ZK forms a right angle with ZQ, because ZK is perpendicular to the plane tangent to the cone and passing through lines AZ and ZQ. Line QZ, however, forms a right angle with ZH. Therefore, ZQ is perpendicular to the plane containing lines HZ and ZK. Moreover, plane QZH intersects plane AZQ along line ZQ, and it intersects line KZ at Z.

[10] Moreover, points T, Z, and K lie to the side of plane HZQ. Therefore, plane HZQ intersects the plane in which DZ and ZK lie. Hence, the common section of planes DZK and HZQ lies in the plane of HZQ. It therefore forms a right angle with ZQ, because ZQ is perpendicular to the plane of HZK, and the common section of these planes lies in between the two lines ZK and ZD. Thus, angle HZD is obtuse, and line HZ lies in the plane that contains lines DZ and ZQ, which is the plane of the [conic] section, and it forms a right angle with ZQ.

[11] Accordingly, if ZH is extended in a straight line from Z, it will cut angle DZK, and line HZ will intersect ED, because it has been demonstrated in the nineteenth proposition of chapter [2] on image[-formation] in book 5 that any two lines perpendicular to two lines tangent to a [conic] section must intersect beyond the arc of the section where the point of contact is. Hence, since line HZ intersects ED, and since HZ cuts angle DZK, line HZ will intersect ED outside of point D, and this is what we wanted [to demonstrate].

LATIN-ENGLISH INDEX

LATIN-ENGLISH INDEX

ENGLISH-LATIN GLOSSARY

ENGLISH-LATIN GLOSSARY

to accord with respicere
actuality veritas
acute acutus
to add addere, multiplicare
to adjoin adiungere
aforesaid predictus
alike/likewise similis / similiter
alternate coalternus
amount quantitas
to analyze declarare, exponere
angle angulus
angle of reflection angulus reflexionis
to appear apparere, videre
appearance apparentia
to apply apponere, aptare
to apply to appropriare
to apprehend apprehendere
apprehension adquisitio
to approach accedere, appropinquare
arc arcus
(area) around circuitus
to arise accidere, evenire, provenire
arrangement ordinatio
to arrive at redere / redire
to assume ponere, proponere, sumere
to augment augmentare
to avoid evitare
axis axis, linea, perpendicularis

base basis
to be accidere
to be apparent/visible apparere
to be clear/evident/obvious certificare, patere
to be compounded constare
to be continuous continuare
to be dropped exire
to be due to provenire
to be larger/longer excedere

to be left/remain remanere
to be less clear latere
to be less than minuere
to be obligated (must) debere
to be opposite diversare
to be parallel equidistare
to be subject to habere
to be tangent to contingere, tangere
to be visible videre
below dimissus
to bisect dividere, secare, transire
blend confusio
to block occultare
body corpus
book liber
bordering conterminabilis
both duplex
boundary extremitas, terminus
breadth latitudo

case (of/for) modus, res
cause causa
to cause efficere, inducere
center/centerpoint centrum, punctum centrum
center of sight centrum, centrum visus, visus
change diversitas
chapter capitulum, pars
to choose/select ponere, sumere
chord corda
circle circulus
circumference circumferentia
to circumscribe describere, revolvere
circumstance situs
clear/evident/plain/obvious manifestus / manifeste, planum
close/near propinquus
closeness accessus
color color
to combine ducere
common communis
common angle angulus communis
common arc communis
common section communis, circulus communis, differentia, differentia communis, linea communis, superficies communis
common term communis
to compound componere, multiplicare
concave concavus, concavitas

concavity concavitas
conclusion coniecturatio
cone piramis
to confuse confundere, dubitare
conic/conical piramidalis
conic section (hyperbola) sectio piramidalis
conical form piramidata
conical shape piramidatio
to connect continuare, copulare
considerable magnus
constraint impedimentum
to construct facere
construction positio
to continue extrahere, transire
contrary contrarius
to converge coniungere, pervenire
converse conversus
convex exterior
to copy iterare
correlation collatio
to correspond respicere
corresponding adinvicem, similis
counterpart compar
cross-section dyameter
curvature arcualitas, concavitas
curved arcualis, tortuosus
to cut abscidere/abscindere, dividere, facere, secare
cylinder columpna
cylindric/cylindrical columpnalis, columpnaris
cylindric section (ellipse) sectio columpnaris
cylindrical mirror columpna

deception deceptio, fallacia
to decrease minuere
degree maioritas
to demonstrate declarare, patere, probare
demonstration demonstratio, probatio
to depend upon facere, respicere
to designate assignare, ponere
designation distinctio
to detect discernere
to determine constituere, determinare
diagram figura, forma
diameter dyameter
to differ diversare, variare
difference proportio, varietas

different inequalis/inequaliter
different types diversitas, varietas
to diminish diminuere, minuere
diminished debilis
diminution minoritas
direct directus, rectus
direct vision directio, directus
direction pars
directly facing rectus
to discern discernere
to discuss declarare, disquirere, loqui
disposition dispositio, situs
to disprove improbare
distance distantia, longitudo, remotio
distant remotus
distinct distinctus
distorted distortus
to divide dividere, partire
division divisio, sectio
to draw continuare, ducere, extrahere, facere, oriri, producere, signare
to drop cadere, ducere, extrahere
to duplicate duplicare, iterare

each case singulum
earlier predictus
end/endpoint caput, extremitas, finis, terminus
to end finire
to ensconce prefigere
entire totalis
entire range universitas
to equal efficere, valere
equality equalitas
equivalent compar
erect/erected erectus, rectus
to erect extrahere
error error
to establish declarare, premittere, preostendere
Euclid Euclides
evident manifestus/manifeste
example exemplum
to exceed egredi, excedere
excessive multiplex
to exist existere
to explain declarare, explanare
explanation explanatio
to extend continuare, ducere, exire, extendere, extrahere, procedere, producere,

protrahere, secare, transire
to extend past excedere
external extrinsecus
to extrapolate extrahere
extreme (means) modus
eye oculus, visus

face facies
to face opponere
facing oppositus
to fall cadere, concurrere
far remotio
far away/from/outside remotus
figure figura
to find invenire, sumere
to finish perficere
flat planus
to follow (logically) procedere, remanere
foregoing predictus
form forma, species
to form continere, efficere, facere, provenire
to form (with respect to) respicere
forming a rectangle ductus...in... (*see also* **rectangle**)

general generalis / generaliter
to get to something accedere
to give dare, proponere, sumere
great magnus
great circle circulus. circulus magnus

half/one-half dimidium, medietas, medius
halfway medius
to happen accidere, evenire, facere
to have habere
having bodily dimensions corporalis
head caput
height altitudo
to hide/make invisible occultare
how (something occurs) modus

identical similis
image forma, ymago
image-location locus ymaginis
image-point punctus ymaginis
to imagine intelligere, proponere
impossibility impossibilitas

impossible impossibilis
incidence accessus
to incline declinare
inclined declinis
inconsiderable modicus
increase additamentum
increase in number pluralitas
infinitude infinitus
infinity infinitum
in place immotus
to intend intendere
intensity fortitudo
intermediate medius
intermediate point punctus medius
intermediate position intermedia
to intersect cadere, concurrere, coniungere, secare
intersection concursus, sectio
inverted conversus

judgment iudicium

key point nomen
to know cognoscere

lack defectus
large magnus
to lead to inducere
to leave be existere
left-hand side sinister
length longitudo, quantitas
less (than) minor/minus
letter littera, nomen
letter-designation littera
to lie cadere, existere, ponere
to lie at a distance/far distare
to lie between interiacere
to lie in front of precedere
to lie outside of declinare, elongare
to lie upon adiacere
light lux
limit finis, terminus
line linea
line of longitude linea longitudinis
line of reflection linea reflexionis
location locum/locus, positio, situs

long longus
longitude longitudo
to look videre
lying above/beyond elevatus
lying far elongatio
lying in front of oppositus

to make premittere, sumere
to make indisputable convincere
manifest manifestus / manifeste
to mark/mark off signare
matter res
mean medius
to measure off mensurare
to meet concurrere
midpoint medius, punctus medius
to mingle miscere
mingling mixtura
mirror speculum
misperception erroneus, fallacia
multitude multitudo

narrow strictus
near(ness) propinquitas
neither neuter
no/not any nullus
normal perpendicularis
not equal inequalis / inequaliter
to note notare
number numerus

object corpus, res
object-point punctum / punctus
oblique obliquus
to obscure occultare
obtuse obtusus
to occlude occultare
to occur accidere, evenire, facere, incidere
on the surface/plane of continuus
opposite contrarius, econversus, oppositus
orientation situs
orientation/respect to pars
to originate oriri
orthogonal/orthogonally ortogonalis / ortogonaliter, perpendicularis / perpendiculariter
other reliquus
outer exterior

overall generalis / generaliter

parallel equidistans
parallelism equidistantia
part pars
particular factor/kind singulum
to pass along/through/to exire, extendere, pertransire, secare, transire
to perceive comprehendere, discernere, percipere
perceptible comprehensibilis
perception comprehensio
periphery circumferentia
perpendicular ortogonalis, perpendicularis
phenomenon res
place locum / locus
to place/replace ponere, proponere
plane planus, superficialis, superficies
to play habere
point locum / locus, punctum / punctus
point of/on a section punctus sectionis
point of division punctus divisionis, punctus sectionis
point of intersection punctus sectionis
point of reflection punctus conversionis, punctus reflexionis
to point out assignare
pole polus
polished tersus
to pose proponere
to posit ponere
position positio, situs
preceding predictus
preliminary points antecedentia
prescribed predictus
to present pretendere
previous predictus
previously discussed/reasoned predictus
to proceed accedere
to produce continuare, ducere, extrahere, facere
to project ducere, facere
pronounced fortis
proof declaratio, demonstratio, probatio
proper disposition veritas
proper orientation rectitudo
proportion proportio
to propose proponere
proposition figura
to prove declarare, probare
to provide premittere

to purpose proponere

radial radialis
radial line linea radialis
radius longitudo
random casualis
range latitudo, quantitas
ratio proportio
to reach attingere, pervenire, venire
to read legere
reason causa, via
reasoning modus
to recapitulate iterare
rectangle ductus...in... (*see also* **forming a rectangle**), multiplicatio
to redraw iterare
to reflect convertere, referre, reflectere, revertere
reflected conversus
reflected vision reflexio
reflecting speculatus
reflection conversio, reflexio
to remain manere
remainder excessus, residuus
remarkable mirabilis
to repeat iterare
reversal conversio
reversed conversus
to revise mutare
right erectus, rectus
right angle angulus rectus
right-hand side dexter
role dignitas
rope funis
to rotate revolvere

sake res
same similis
section linea, portio, sectio, sector
to see videre
segment pars, portio
semicircle semicirculus
to separate separare
separation distantia
to set forth/out premittere, proponere
several/several times larger multiplus
shape figura, forma
short brevis

to shorten diminuere
to show declarare, ostendere, patere, preostendere
side latus, pars
sight visus
significant magnus
similar/similarly similis / similiter
situation positio, situs
size magnitudo, quantitas
slant declinatio
to slant declinare
slanted declinis, obliquus
slanted line linea declinationis
slight modicus, parvus
small minor / minus, modicus, parvus
solid corporalis
solid angle angulus corporalis
some modicus
soul anima
spatial disposition situs
specific specialis
sphere spera
spherical spericus
square (x^2) quadratum
start initium
stationary immobilis
to stipulate ponere
straight rectus
straight line linea recta, rectitudo
straightness rectitudo
strong fortis
sub-angle distinctio, particula
to subtend respicere
to subtract aufere, subtrahere
subtraction ablatio
to suppose ponere
surface superficies

to take ponere, sumere
to take the shape of assimulare
tangency contingentia
tangle intricatio
terminal terminus
terminal segment caput
theorem figura
thing res
to think about intelligere

threshold temperantia
threshold condition temperamentum
time hora
tiny feature minutia
token modus
to touch cadere, secare, tangere
train (of logic) via
triangle triangulus

to understand intelligere
upright erectus, rectus

variation diversitas, variatio
variety varietas
to vary diversare
to verge towards accedere
vertex acumen, caput
vertical (angle) collateralis, contrapositio
vice-versa econversus
to view videre
viewer aspiciens, visus
visible visibilis
visible object(s) res visa, visibilia, visibilis, visum
visible point punctus visus
vision visus
visual visualis
visual axis axis, axis visualis, linea visualis
visual faculty visus
visual sense sensus

to want volere
way dispositio, modus
weak debilis
to weaken debilitare
weakening debilitas, debilitatio
whole totalis
width latitudo
window foramen
written text scriptus

to yield habere

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Earlier books on Alhacen's optics by A. Mark Smith

Alhacen's Theory of Visual Perception: A Critical Edition, with English Translation and Commentary, of the First Three Books of Alhacen's *De Aspectibus*, the Medieval Latin Version of Ibn al-Haytham's *Kitāb al-Manāzir*—Transactions of the American Philosophical Society, volume 91, parts 4 and 5 (ISBN: 0-87169-914-1)

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